# Nonholonomic dynamics, optics and the least time.

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with A. Rojo

- **Least Action**
- **Optical Mechanical Analogy**
- **Hamiltonization**
- **Nonlinear Constraints**

Well known that there is an analogy between optics and mechanics that inspired much of the classical theory of mechanics and indeed extended to the theory of quantum mechanics.

Here we develop the optical mechanical analogy for a prototypical non-holonomic mechanical system (a system with non-integrable velocity constraints): a knife-edge moving on the plane subject to a potential force.

**Nonholonomic systems are not Hamiltonian or indeed variational so this analogy is quite subtle. There is an interest going back to the work of Chaplygin in finding a time transformation that "Hamiltonizes" a (reduced) nonholonomic system (see also the work of Federov and Jovanovic, Borisov and Mamaev and Fernandez, Mestdag and Bloch. See also work with Zenkov.**

**We show that our analysis provides a somewhat different approach to this idea. A key in all our analysis is to note that the time variable is changed and so while trajectories are mapped to trajectories the dynamics along the trajectories change. Also important is the role played by gyroscopic forces and the gyroscopic-like terms in the nonholonomic equations.**

**A key difference in our analysis is that our time change is dependent on the trajectory. We normally choose zero energy, as in the classical analysis of the knife edge on the plane, but our treatment can be extended to arbitrary energy without loss of generality. In this sense our analysis is closer the principle of least action than to the Lagrange D'Alembert principle, which is of course the standard approach to nonholonomic systems and is fundamental also to the Chaplygin Hamiltonization.**

**We explore various potentials for the nonholonomic system that gives rise to classical dynamic orbits in the plane and derive the associated index of refraction for the corresponding optical system.**

**The general equations of motion for a nonholonomic system may be formulated as follows. Let $Q$, a smooth manifold, be the configuration space of the system. Let $\{\omega^a\}$ be a set of $m$ independent one-forms whose vanishing describes the constraints on the system; that is, the constraints on system velocities are defined by the $m$ conditions $\omega^a \cdot v = 0$, $a = 1, \ldots, m$. Using the fact that these $m$ one-forms are independent one can choose local coordinates such that the one-forms $\omega^a$ have the form**

$$\omega^a(q) = ds^a + A^a_\alpha(r, s)dr^\alpha, \qquad a = 1, \ldots, m, \tag{2.1}$$

**where $q = (r, s) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$.**

**With this choice, the constraints on virtual displacements (variations)** $\delta q = (\delta r, \delta s)$ **are given by the conditions**

$$\delta s^a + A^a_\alpha \delta r^\alpha = 0. \tag{2.2}$$

**Now the Lagrange-D'Alembert principle gives the equations**

$$-\delta L = \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0, \tag{2.3}$$

**for all variations** $\delta q$ **such that** $\delta q$ **that satisfy the constraints.**

**Substituting (2.2) into (2.3) and using the fact that** $\delta r$ **is arbitrary gives**

$$\left( \frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} \right) = A^a_\alpha \left( \frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a} \right), \quad \alpha = 1, \ldots, n-m. \tag{2.4}$$

**The equations** (2.4) **combined with the constraint equations**

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha, \quad a = 1, \ldots, m, \tag{2.5}$$

**give a complete description of the** *equations of motion* **of the system. Notice that they consist of** $n-m$ **second-order equations and** $m$ **first-order equations.**

**We now define the "constrained" Lagrangian by substituting the constraints (2.5) into the Lagrangian:**

$$L_c(r^\alpha, s^a, \dot{r}^\alpha) = L(r^\alpha, s^a, \dot{r}^\alpha, -A^a_\alpha(r,s)\dot{r}^\alpha).$$

**The equations of motion (2.4) can be written in terms of the constrained Lagrangian in the following way, as a direct coordinate calculation shows:**

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta}\dot{r}^\beta, \tag{2.6}$$

**where**

$$B^b_{\alpha\beta} = \left( \frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a} \right). \tag{2.7}$$

**Now one can show that the system is holonomic if and only if the the coefficients (2.7) vanish. More generally the system is Lagrangian if the right hand side of (2.6) vanishes. One can view the goal of Hamiltonization as finding a change in the time variable such that this occurs.**

# Examples:

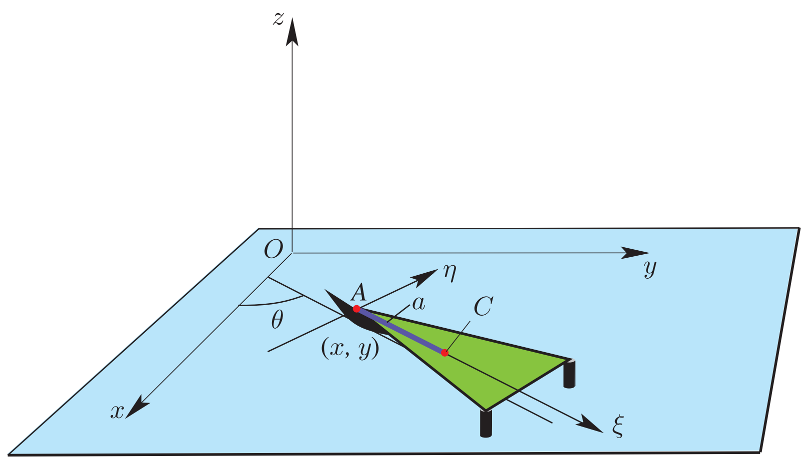


Figure 2.1: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

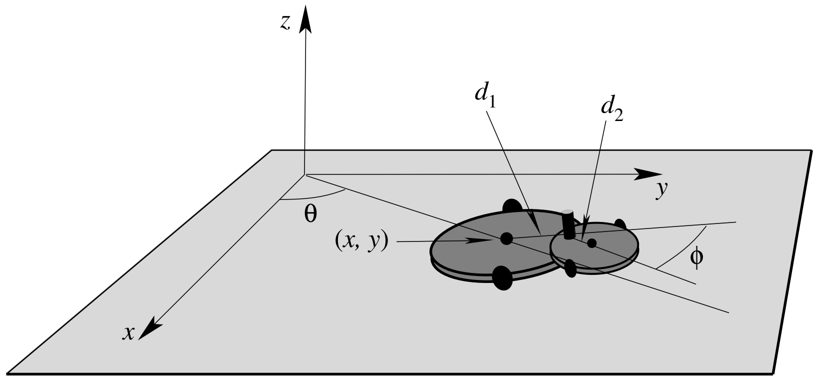


Figure 2.2: The geometry for the roller racer.

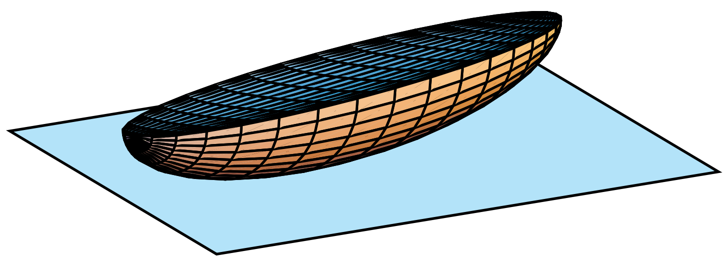

Figure 2.3: The rattleback.

# Knife Edge on Inclined Plane:

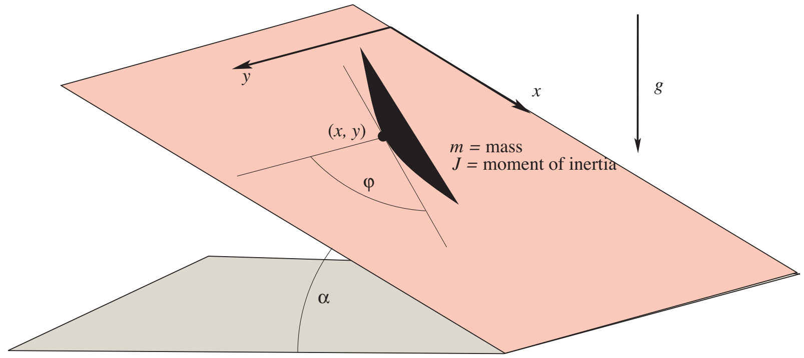


Figure 2.4: Motion of a knife edge on an inclined plane.

**The knife edge Lagrangian is taken to be**

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\varphi}^2 + mgx\sin\alpha \tag{2.8}$$

**with the constraint**

$$\dot{x}\sin\varphi = \dot{y}\cos\varphi\,. \tag{2.9}$$

**The equations of motion:**

$$m\ddot{x} = \lambda \sin\varphi + mg\sin\alpha\,,$$
$$m\ddot{y} = -\lambda\cos\varphi\,,$$
$$J\ddot{\varphi} = 0\,.$$

**We assume the initial data** $x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = \varphi(0) = 0$ **and** $\dot{\varphi}(0) = \omega$**. The energy:**

$$E = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\varphi}^2 - mgx\sin\alpha$$

**and is preserved along the flow. Since it is preserved, it equals its initial value**

$$E(0) = \frac{1}{2}J\omega^2\,.$$

**Hence, we have**

$$\frac{1}{2}\frac{\dot{x}^2}{\cos^2\varphi} - mgx\sin\alpha = 0\,.$$

**Solving, we obtain**

$$x = \frac{g}{2\omega^2}\sin\alpha\sin^2\omega t$$

**and, using the constraint,**

$$y = \frac{g}{2\omega^2} \sin\alpha \left( \omega t - \frac{1}{2} \sin 2\omega t \right) .$$

**Hence the point of contact of the knife edge undergoes a cycloid motion along the plane, but does not slide down the plane.**

**Different from the vakonomic (variational) sleigh (Kozlov, Arnold...).**

**We now consider our prototypical example, the knife edge. We develop first the geometry of the trajectories of a knife edge moving on the plane co-ordinatized by $(x, y)$ with blade angle $\theta$ with the $y$-axis.**

**Call $\phi(s)$ the tangent angle to a the trajectory [and $s$ the arc length, not to be confused with the variable $s$ in the pair $(r, s)$ of the previous section] of a particle moving in two dimensions**

$$\mathbf{x}(s) = \left( \int_0^s ds \sin\phi(s), \int_0^s ds \cos\phi(s) \right), \tag{3.1}$$

**and**

$$(\dot{x}, \dot{y}) = \dot{s} \left( \sin\phi(s), \cos\phi(s) \right). \tag{3.2}$$

**From which follows the relation**

$$\dot{x}\cos\phi - \dot{y}\sin\phi = 0, \tag{3.3}$$

**valid *for any* unconstrained curve. Imposing a knife edge constraint amounts to imposing the equality between the tangent angle to the curve, and the knife edge angle $\theta$, which in principle is independent of $\phi$.**

**In other words, if**

$$\theta = \phi \tag{3.4}$$

**then we have the knife edge in the usual form:**

$$\dot{x}\cos\theta - \dot{y}\sin\theta = 0. \tag{3.5}$$

**Once the constraint is imposed we can analyze the properties of the center of mass motion parametrizing the curve in terms of the arc length $s$ and the tangent $\theta(s)$ to the curve:**

$$\theta = \theta(s). \tag{4.1}$$

**We restrict to the case of "free" knife edge motion, meaning that the knife edge variable $\theta$ is not subject to a $\theta$ dependent potential. Now, for a knife edge constraint we have**

$$\dot{\theta} = \omega\,. \tag{4.2}$$

**and with $\omega$ a constant we obtain:**

$$\omega = \frac{d\theta(s)}{ds}\frac{ds}{dt} = \frac{1}{\rho(\mathbf{x})}\frac{p(\mathbf{x})}{m}, \tag{4.3}$$

**where $\rho$ is the radius of curvature of the curve and $p$ the mo-**

**mentum of the particle (we are considering fixed energy since the system conserves energy). So, a knife edge trajectory satisfies a simple relation between the radius of curvature and the momentum:**

$$\rho(\mathbf{x}) = \frac{1}{\omega m} p(\mathbf{x}) \tag{4.4}$$

**Using the constraint in the form $\dot{x} = \dot{y}\tan\theta$ the reduced Lagrangian (with the constraint substituted) and mass equal to unity becomes**

$$L_c = \frac{1}{2}(\dot{y}^2 \sec^2\theta + \dot{\theta}^2) - V \tag{5.1}$$

**In this case, in the notation above the variable $s$ is equal to $x$ and the variables $(y, \theta)$ are the $r$-variables. Assume for simplicity that $V$ is independent of $x$.**

**In the constraint equation (2.5) we thus have $A_1^1 = -\tan\theta$ while $A_2^1 = 0$.**

**Thus**

$$B_{12}^1 = -B_{21}^1 = \frac{\partial A_1^1}{\partial \theta} - \frac{\partial A_2^1}{\partial y} = \sec^2\theta \,. \tag{5.2}$$

**Then the dynamic equations of motion for $y$ and $\theta$ follow from (2.6) and are given respectively by**

$$\ddot{y}\sec^2\theta + 2\dot{y}\sec^2\theta\tan\theta\dot{\theta} \;=\; \dot{x}\sec^2\theta\dot{\theta} - \frac{\partial V}{\partial y} = \dot{y}\dot{\theta}\sec^2\theta\tan\theta - \frac{\partial V}{\partial y}$$

$$\ddot{\theta} - \dot{y}\dot{\theta}\tan\theta\sec^2\theta \;=\; -\dot{x}\sec^2\theta\dot{\theta} - \frac{\partial V}{\partial \theta} = -\dot{y}\dot{\theta}\sec^2\theta\tan\theta - \frac{\partial V}{\partial \theta}$$

**where we used the constraints.**

**Hence we obtain**

$$\ddot{y} \quad = \quad -\dot{y}\dot{\theta}\tan\theta - \cos^2\theta\frac{\partial V}{\partial y} \tag{5.3}$$

$$\ddot{\theta} = - \frac{\partial V}{\partial \theta}\,. \tag{5.4}$$

**This, together with the constraints, defines the dynamics.**

**As an example, consider** $V = 0$**, and** $\dot{\theta} = \omega$**, where the above equations, together with the constraint imply:**

$$\begin{aligned} \ddot{y} &= -\omega\dot{x} \\ \ddot{x} &= +\omega\dot{y}, \end{aligned} \tag{5.5}$$

**which corresponds to the knife edge moving in a circular orbit and rotating at angular velocity** $\omega$

**The classical optical mechanical analogy stems from the isomorphsim between trajectories of a particle of mass $m$, moving at constant energy $E$ in a potential $V(\mathbf{x})$ (the momentum being $p(\mathbf{x}) = \sqrt{2m(E - V(\mathbf{x})})$, and that of a light ray that propagates, at constant frequency, in a medium of index of refraction $n(\mathbf{x})$. In each case, if $x_i$ and $x_f$ are the initial and final points, the trajectories are the extrema of their corresponding action functionals:**

$$\begin{aligned} S_o &= \int_{x_i}^{x_f} nds \quad \text{(geometric optics)} \\ S_m &= \int_{x_i}^{x_f} pds \quad \text{(mechanics).} \end{aligned} \tag{6.1}$$

**The analogy results from the equivalence of two conservation laws: conservation of momentum in the direction parallel to the surfaces of constant potential (Newton's second law for particles) and conservation of wave vector (or "slowness") in the direction parallel to the surfaces of constant index of refraction (Snel's law for light rays).**

**The analogy implies that the physical trajectories between $x_i$ and $x_f$ can be either computed for a light ray or for a particle, provided one has the equivalence**

$$p(\mathbf{x}) = \sqrt{2m(E - V(\mathbf{x})} = n(\mathbf{x}). \tag{6.2}$$

**Notice that $p$ and $n$ have different units, but this is irrelevant in determining the geometry of the trajectories since the respective units amount to multiplicative constants in their actions.**

**The optical mechanical analogy elevated its status with the advent of quantum mechanics, and the early search of a wave mechanics for particles. The natural question is: if geometric optics is the small wave length limit of wave optics, what plays the role of a wave length $\lambda$ for particles, in such a way that Newtonian mechanics is recovered in the limit of small $\lambda$? The optical mechanical analogy provides the natural correspondence:**

$$p(\mathbf{x}) \propto n(\mathbf{x}) \propto \frac{1}{\lambda(\mathbf{x})}. \tag{6.3}$$

**Since $p$ and $\lambda$ have different units there must be a constant of proportionality between them: $p = h/\lambda$, the celebrated De Broglie's relation, with the proportionality constant (Planck's universal constant) determined experimentally.**

**Now we explore an extension of the optical mechanical analogy to a nonholonomic system. The special interest of this problem results from the fact that the nonholonomic trajectories are not determined by a Least Action Principle, so the analogy in principle does not apply in the usual sense.**

**Consider the trajectory of a light ray propagating in an arbitrary two–dimensional index of refraction $n(\mathbf{x})$, and choose $n$ such that the local curvature of the ray $\kappa = 1/\rho$ is precisely that of the trajectory of the knife edge, as given by Eq. (4.4). This step leads us, following Hamilton's program of the optical mechanical analogy, to an equivalent Hamilton-Jacobi equation for a non-holonomic system.**

**The problem of the curvature of a light ray in an arbitrary index of refraction was treated by Born and Wolf in their classic "Principles of Optics". Here we re-derive the same result using a slightly different approach for completeness. We discretize the problem into lines of constant $n$, as in Figure (6.1).**

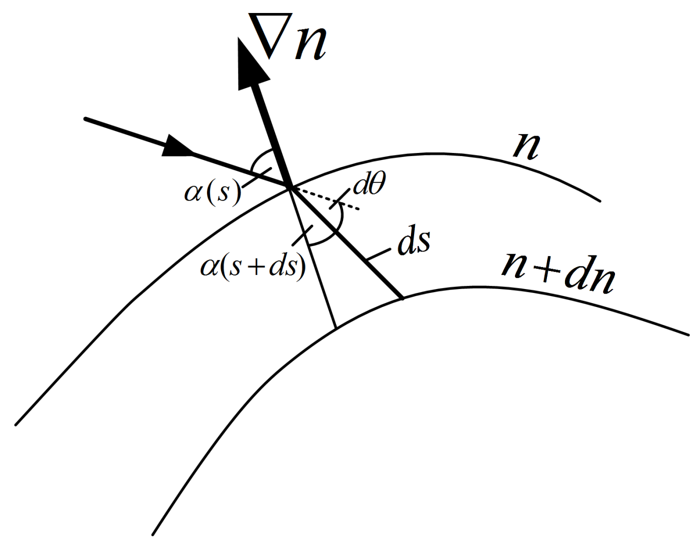


Figure 6.1: Discretization of the trajectory of a light ray in a spatially dependent index of refraction $n$

**Snel's law for a ray refracting on one of this lines is**

$$n(s)\sin\alpha(s) = n(s+ds)\sin\left(\alpha + d\theta\right), \tag{6.4}$$

**where $\alpha(s)$ is the angle the light ray makes with the normal to the surface of constant $n$, $s$ is the arc length and $d\theta$ is the change of the angle of the tangent to the curve [See Figure (6.1)]. Notice that in this general case, whereas $\theta$ is the angle that the tangent makes with a fixed direction in space, $\alpha$ is the angle that the tangent makes with the gradient of $n$.**

**Now expand the right hand side of Equation (6.4) to obtain**

$$\frac{d\theta}{ds} = -\frac{n'(s)}{n(s)}\tan\alpha(s). \tag{6.5}$$

**Since $\alpha$ is the angle of the tangent to the ray with the normal to the surface,**

$$\frac{dn(s)}{ds}\frac{1}{\cos\alpha(s)} = |\nabla n|, \tag{6.6}$$

**and from this equation we obtain the general expression for the curvature of the light ray**

$$\frac{d\theta}{ds} \equiv \kappa(s) = -\frac{|\nabla n|}{n(s)}\sin\alpha(s), \tag{6.7}$$

**which leads us to the optical mechanical analogy for the knife edge, relating the translational momentum $p$ (a quantity determined by the local potential at constant energy), and the index of refraction:**

$$\boxed{\frac{\omega m}{p(s)} = -\frac{|\nabla n|}{n(s)}\sin\alpha(s).} \tag{6.8}$$

**This equation is the main result, relating geometric optics with nonholonomic mechanics. Notice the difference with the usual optical mechanical analogy, for which $p = n$.**

**In order to apply the optical mechanical analogy we need to find, explicitly, $n$ given $p$, and this we were able to do for cases where there is a constant of motion that relates $\sin\alpha(s)$ with position.**

**Example: $n(x, y) = n(y)$ and the brachistochrone**

**Here we consider the case where the index of refraction varies in one of the spatial directions only, as is the case for models of mirages and in the brachistochrone, one of the paradigmatic variational problems. For this case $|\nabla n| \sin\alpha = (dn/dy) \sin\theta$ (we put $\sin\alpha = \sin\theta$ since the normal to $n$ has a constant direction in space). Also, Snel's Law in this case gives $n \sin\theta = C$, with $C$ a constant, and we can integrate (6.8) to obtain**

$$n(y) \propto \left( \int \frac{dy}{p(y)} \right)^{-1} . \tag{7.1}$$

**As a particular example consider the case of the knife edge falling on an inclined plane (with potential proportional to $y$), for which (at zero translational energy) $p(y) = a\sqrt{y}$, with $a$ a constant. Substituting in (7.1) we obtain**

$$n(y) = \frac{c}{\sqrt{y}}, \tag{7.2}$$

**and this corresponds to the classical mapping proposed first by Bernoulli between the brachistochrone, a minimization of *time* problem, and the motion of a light ray. The point pertinent to our present treatment is that the brachistochrone trajectory is a cycloid–that is, the motion of a light ray moving in an index of refraction such as that of Eq. (7.2) is a cycloid–and so is the motion of a knife edge in an inclined plane.**

**Example: Constant $n$**

**Consider the case $p = \text{constant}$. For a knife edge the motion corresponds to circular motion. According to (7.1) this would correspond to an index of refraction**

$$n(y) \propto \frac{1}{y}. \tag{7.3}$$

**The trajectory of a light ray with an index of refraction with this functional dependence is given by Snell's law**

$$\begin{aligned} n(y) \sin\theta &= C, \\ \sin\theta &= Cy, \end{aligned} \tag{7.4}$$

**which of course is the equation of a circle.**

**Example: Rotational Symmetry, $n(x,y)=n(r)$**

**In the case of a central potential we make use of the Formula of Bouguer –the conservation of angular momentum for light rays:**

$$rn(r)\sin\alpha(r)=L \;\; \text{(Constant)}, \tag{7.5}$$

**and the curvature of a light ray in a central potential, substituting (7.5) in (6.8) is**

$$\kappa(s)=\frac{d\theta(s)}{ds}=-\frac{1}{r}\frac{dn}{dr}\frac{L}{n^2} \tag{7.6}$$

**Using Equation (4.4) we can invert (7.6) to obtain**

$$\frac{1}{n(r)}\propto\int\frac{r\,dr}{p(r)} \tag{7.7}$$

**Example: Logarithmic spiral**

**In order to exemplify our treatment for central potentials we proceed in reverse. We consider a few curves for which the curvature as a function of the radius can be computed easily, and derive the potential for which a knife edge will describe the curve in question. Consider the curve of the logarithmic spiral**

$$r(\theta) = ae^{b\theta} \tag{7.8}$$

**In order to find the corresponding $p(r)$ for the knife edge we compute the curvature $\kappa$. In cylindrical coordinates the curvature is given by the following relation:**

$$\kappa = \frac{|r^2 + 2r'^2 - rr''|}{\left(r^2 + r'^2\right)^{3/2}}, \tag{7.9}$$

**which, for the logarithmic spiral gives**

$$\kappa = \frac{1}{\sqrt{2}r}, \quad R = \sqrt{2}r. \tag{7.10}$$

**Using (4.4),**

$$p(r) = \sqrt{2}\omega m r, \quad V(r) = -m\omega^2 r^2, \tag{7.11}$$

**a repulsive quadratic potential. In other words, given $\omega$ as a constant of motion, the knife edge "falls" in a repulsive harmonic potential following a logarithmic spiral. Now let's see how this connects with the corresponding optical problem.**

**According to our mapping (7.7), the corresponding index of refraction is, in this case,**

$$n(r) = \frac{C}{r}, \tag{7.12}$$

**with $C$ a constant.**

**Using this dependence in the Formula of Bouguer, Eq. (7.5), we obtain**

$$\sin\alpha(r) = LC, \tag{7.13}$$

**which means that the trajectory of a light ray in an index of refraction** $n = C/r$ **forms a constant angle of incidence with the surfaces of constant** $n$**. This is a well known property of the logarithmic spiral: the tangent to the curve forms a constant angle with the radius vector.**

**We illustrate this with a simple calculation:**

$$\mathbf{r} = ae^{b\theta(t)}\hat{\mathbf{u}}_r. \tag{7.14}$$

**The tangent vector to the spiral is therefore**

$$\hat{\mathbf{t}} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{b\hat{\mathbf{u}}_r + \hat{\mathbf{u}}_\theta}{\sqrt{1+b^2}}. \tag{7.15}$$

**which means that, indeed, for this curve the tangent forms a constant angle with $\hat{\mathbf{u}}_r$,**

$$\sin\alpha = \frac{1}{\sqrt{1+b^2}}, \tag{7.16}$$

**the classic property of the logarithmic spiral We see that the parameter $b$ is related to the "angular momentum" $L$ of the ray. For $L = 0$, $b \to \infty$ we get a straight line. for $b = 0$ we obtain a circle, which is also an orbit for the knife edge.**

**It is instructive to consider this dynamics from the standard nonholonomic point of view, enforcing the constraint by Lagrange multipliers.**

**Consider the knife edge in a repulsive potential**

$$V(r) = -\frac{1}{2}m\omega_0^2 r^2 \tag{7.17}$$

**The non-holonomic equations of motion are**

$$\begin{aligned} \ddot{x} &= \omega_0^2 x - 2\frac{\lambda}{m}\cos\theta(t) \\ \ddot{y} &= \omega_0^2 y + 2\frac{\lambda}{m}\sin\theta(t), \end{aligned} \tag{7.18}$$

**where time dependent Lagrange multipliers are introduced in the equations of motion to enforce the constraint.**

**We want to check that the parametric equations for a logarithmic spiral, given by**

$$\begin{aligned} x(t) &= r_0 e^{bt} \cos \alpha t \\ y(t) &= r_0 e^{bt} \sin \alpha t, \end{aligned} \tag{7.19}$$

**are indeed solutions of the non-holonomic equations. The initial conditions for the spiral (for zero translation energy of the particle)**

$$\begin{aligned} \frac{1}{2} m \dot{\mathbf{x}}^2 &= \frac{1}{2} m r_0^2 (\alpha^2 + b^2) \\ &= \frac{1}{2} m \omega_0^2 r_0^2, \end{aligned} \tag{7.20}$$

**which implies that the parameters of the spiral satisfy:**

$$\alpha^2 + b^2 = \omega_0^2. \tag{7.21}$$

**Also, the tangent vector to the spiral is**

$$\begin{aligned}\hat{\mathbf{t}} &= \frac{(b\cos\alpha t - \alpha\sin\alpha t, b\sin\alpha t + \alpha\cos\alpha t)}{\sqrt{b^2+\alpha^2}} \\ &= (\cos(\alpha t+\delta), \sin(\alpha t+\delta))\end{aligned} \tag{7.22}$$

**with**

$$\cos\delta = b/\sqrt{b^2+\alpha^2} = b/\omega_0.$$

**Since the tangent angle is rotating at a constant rate, we have**

$$\alpha = \omega$$

**with** $\alpha$ **the initial angular velocity of the knife edge–a constant of motion–and where** $\delta$ **is the initial angle that the knife edge makes with the** $x$ **axis.**

**Now we compute the second derivative of the spiral equation (Eq. (7.19))**

$$
\begin{aligned}
\ddot{x} &= (b^2 - \omega^2)x - 2b\omega r_0 e^{bt} \cos \omega t \\
&= (b^2 + \omega^2)x - 2\omega r_0 e^{bt} \left(b \cos \alpha t - \omega \sin \omega t\right) \\
&= \omega_0^2 x - 2 r_0 e^{bt} \omega \sqrt{\omega^2 + b^2} \cos(\omega t + \delta) \\
&= \omega_0^2 x - \frac{\lambda}{m} \cos \theta(t), \qquad (7.23)
\end{aligned}
$$

**and, similarly**

$$
\ddot{y} = \omega_0^2 y + \frac{\lambda}{m} \sin \theta(t), \qquad (7.24)
$$

**with**

$$
\begin{aligned}
\theta(t) &= \omega t + \delta \\
\lambda &= 2m2r_0 e^{bt} \omega \sqrt{\omega^2 + b^2} \,. \qquad (7.25)
\end{aligned}
$$

**In summary, we have verified that, if a knife edge in a repulsive potential starts (at zero translational kinetic energy) with initial conditions $r = r_0$, $\omega$ for the angular frequency of the knife, and $\delta$ for the initial orientation of the velocity (and the edge), the corresponding motion is a logarithmic spiral given by**

$$
\begin{aligned}
x(t) &= r_0 e^{\omega_0 t \cos\delta} \cos\omega t \\
y(t) &= r_0 e^{\omega_0 t \cos\delta} \sin\omega t
\end{aligned}
\qquad (7.26)
$$

#### 7.0.1 Lemniscate of Bernoulli

**This curve is described in polar coordinates as**

$$r^2(\phi) = 2a^2 \cos 2\phi, \tag{7.27}$$

**and has the interesting property that the radius of curvature varies as the inverse of the polar radius:**

$$R = \frac{2a^2}{3r}. \tag{7.28}$$

**According to the basic mapping of Eq. (4.4), the knife edge problem has this orbit for $p \propto 1/r$, or for a potential**

$$V(r) \propto -\frac{1}{r^2}, \tag{7.29}$$

**(an attractive potential that decreases as the inverse square power) and zero total energy.**

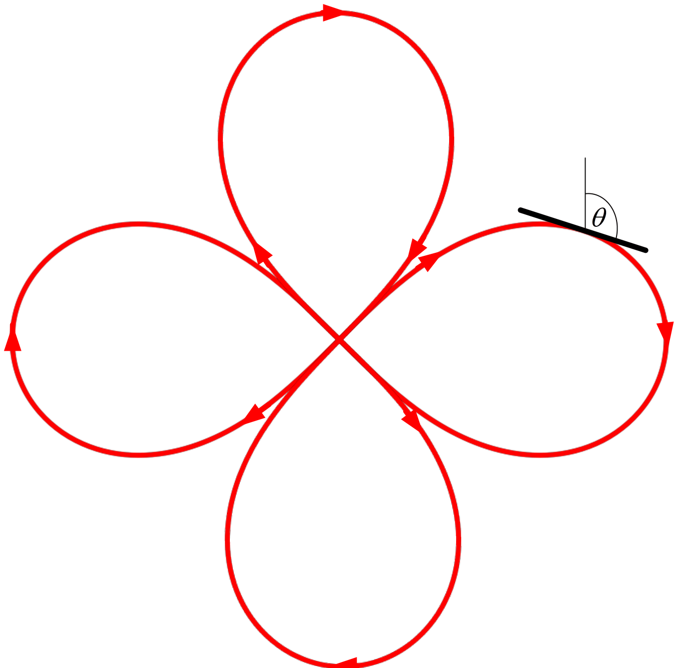


Figure 7.1: Lemniscate of Bernoulli. Notice that Eq. (7.27) describes a curve with two "lobes" that in principle could be oriented in any direction in the plane. Since the direction of rotation is continuous, the motion of the knive edge will describe four lobes, corresponding to the two lemniscates shown in the figure. If the knife edge is rotating clockwise, it will transition from lobe to lobe counterclockwise.

**The corresponding index of refraction for this problem is**

$$n \propto \frac{1}{r^3} \tag{7.30}$$

**Finally, we verify that indeed the lemniscate motion corresponds to a *holonomic* potential $\bar{V} \propto n^2$, which in this case is $\bar{V} \propto -1/r^6$.**

**In general, for a particle moving in a central potential we have**

$$\frac{dr}{d\theta} = \sqrt{\frac{2mEr^4}{L^2} - r^2 - \frac{2mr^4V(r)}{L^2}} \tag{7.31}$$

**For** $V = -\alpha r^{-6}$ **and** $E = 0$

$$\frac{dr}{d\theta} = \sqrt{\beta - r^4}/r \tag{7.32}$$

**with** $\beta = 2m\alpha/L^2$**, and which is obeyed, as expected by,**

$$r^2(\theta) = \beta \sin 2\theta. \tag{7.33}$$

**We have so far presented the optical mechanical analogy and illustrated its applicability in several specific examples. Upon mapping the problem to one of geometric optics (where the Least Action Principle applies) we are in fact converting the problem into a Lagrangian or Hamiltonian problem. This is possible only because we are concerned with the geometry of the trajectories and not with their specific dynamics. In the following sections we describe a related approach by Chaplygin, who "Hamiltonizes" the problem through a reparametrization of time.**

**The simplest setting is when the constraint functions $A^a_\alpha$ and $L$ are independent of $s$, in which case the last term on the left hand side of equation (2.6) vanishes as do the last two terms of (2.7).**

**For the classical case of Chaplygin Hamiltonization we assume that there are only two base variables $r^1$ and $r^2$ and that the $A^a_\alpha$ depend only on these variables.**

**The constraints take the form**

$$\dot{s}^a = -A^a_1 \dot{r}^1 - A^a_2 \dot{r}^2, \; a = 1 \ldots m \,. \tag{8.1}$$

**In this case we can compute that the equations (2.6) become**

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^1} - \frac{\partial L_c}{\partial r^1} \; = \; \dot{r}^2 S \tag{8.2}$$

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^2} - \frac{\partial L_c}{\partial r^2} \; = \; -\dot{r}^1 S \tag{8.3}$$

**where**

$$S = -\frac{\partial L}{\partial \dot{s}^b}\left(\frac{\partial A_1^b}{\partial r^2} - \frac{\partial A_2^b}{\partial r^1}\right) . \tag{8.4}$$

**Our goal is to make these equations Lagrangian.**

**To this end change to the new time variable**

$$d\tau = N(q)dt \, . \tag{8.5}$$

**Denote the derivative with respect to new time variable as primed, i.e.** $\dot{q}^i = N(q)q^{'i}$**. Also, denote** $L_c$ **in terms of this time variable by** $\overline{L}_c$**.**

**Then we have**

$$\frac{\partial L_c}{\partial \dot{r}^\alpha} = \frac{1}{N}\frac{\partial \overline{L}_c}{\partial r'^\alpha} \tag{8.6}$$

$$\frac{\partial L_c}{\partial r^\alpha} = \frac{\partial \overline{L}_c}{\partial r^\alpha} - \frac{1}{N}\frac{\partial N}{\partial r^\alpha}\sum_{\alpha=1}^{2} r'^\alpha \frac{\overline{L}_c}{\partial r'^\alpha} \tag{8.7}$$

**Then a computation shows that the equations (8.3) become**

$$\frac{d}{d\tau}\frac{\partial \overline{L}_c}{\partial r'^1} - \frac{\partial \overline{L}_c}{\partial r^1} \;=\; r'^2 R \tag{8.8}$$

$$\frac{d}{d\tau}\frac{\partial \overline{L}_c}{\partial \dot{r}'^2} - \frac{\partial \overline{L}_c}{\partial r^2} \;=\; -r'^1 R\,, \tag{8.9}$$

**where**

$$R = NS - \frac{1}{N}\left(\frac{\partial N}{\partial r^2}\frac{\partial \overline{L}_c}{\partial r'^1} - \frac{\partial N}{\partial r^1}\frac{\partial \overline{L}_c}{\partial r'^2}\right) \tag{8.10}$$

**Hence if we can choose $N$ such that $R$ is zero we have reduced the equations to Lagrangian and hence Hamiltonian form. Further generalizations are possible.**

**Now let us return to the knife edge.**

**The reduced Lagrangian was written in (5.1) (and may be generalized in this Chaplygin setting to include any potential which does not depend on $x$).**

**As above the classical Chaplygin Hamiltonization proceeds by introducing a time change of the form $d\tau = N(q)dt$ which makes the reduced dynamics Hamiltonian or Lagrangian.**

**Here we can show that $N = \cos\theta$ satisfies the Hamiltonization condition, i.e. sets R, equation (8.10), to zero. Thus the derivative with respect to $\tau$ of a variable $q$, which we will denote $q'$ is related to that with respect to $t$ by**

$$\dot{q} = q' \cos\theta, . \tag{9.1}$$

**Setting the potential equal to zero for convenience and using the constraint in the form $\dot{x} = \dot{y}\tan\theta$ the reduced kinetic energy (with the constraint substituted) and mass equal to unity becomes**

$$T = \frac{1}{2}(\dot{y}^2 \sec^2\theta + \dot{\theta}^2) \tag{9.2}$$

**while T in the $\tau$-time becomes**

$$T = \frac{1}{2}(y'^2 + \theta'^2 \cos^2\theta)\,. \tag{9.3}$$

**The Lagrangian equations in the $\tau$-time are thus**

$$\begin{aligned} y'' &= 0 && (9.4) \\ \theta'' &= (\theta')^2 \tan\theta\,. && (9.5) \end{aligned}$$

**Now to see that these are the correct nonholonomic equations replace the $\tau$ derivatives by the derivatives with respect to $t$ and we obtain**

$$\ddot{y} = -\dot{y}\dot{\theta}\tan\theta \tag{9.6}$$

$$\ddot{\theta} = 0\,. \tag{9.7}$$

**which are the nonholonomic equations which one can supplement with the constraint giving the dynamics in $x$. It is then possible to introduce any potential function which depends on $y$ and $\theta$.**

**In order to make the connection between the Chaplygin treatment and ours, let $\tau$ be the time parameter of our "Hamiltonized" problem (using the optical analogy) and $t$ that of original nonholonomic knife edge**

$$\frac{ds}{d\tau} \equiv \frac{p_H}{m} = an, \tag{9.8}$$

**with $a$ a constant, and $n$ the index of refraction of the optical mechanical problem, and $p_H$ is the momentum of the Hamiltonized, nonholonomic particle. For an index of refraction that depends on one coordinate $n = n(y)$ we can relate $n$ to $\theta$ using Snel's law $n \cos\theta = const$, where $\theta$ is the angle with respect to the $x$ axis:**

$$\frac{ds}{d\tau} = \frac{A}{\cos\theta}, \tag{9.9}$$

**with $A$ a constant. For the true dynamics, on the other hand, we have:**

$$\frac{ds}{dt} = \frac{p}{m}. \tag{9.10}$$

**The time parameters are therefore related through**

$$d\tau = Cpdt\cos\theta, \tag{9.11}$$

**and the parametrizations coincide when $p$ is constant (no potential).**

**Magnetic Analogy:**

**Present here another mapping of the trajectories of constant** $E$ **to a holonomic system, and show that the trajectories are equivalent to those of a spatially dependent magnetic field with no external potential.**

**We can map the knife edge trajectories to those of a particle in a magnetic field as follows:**

**A particle of unit mass and unit charge, moving in a spatially dependent magnetic field** $B(\mathbf{x})$ **, and in the absence of a potential, has a local radius of curvature** $\rho = v_0/B$**, with** $v_0$ **a constant of the motion. This means that the trajectory of a knife edge of translational kinetic energy** $E$ **and moving in a potential** $V(\mathbf{x})$ **is equivalent to that of a particle of velocity** $v_0$ **in a magnetic field given by**

$$B(\mathbf{x}) = \frac{v_0 m\omega}{p(\mathbf{x})} \tag{9.12}$$

**In this case $V(\mathbf{x}) = -\alpha^2 y/2$ and we treat the case of $E = 0$, for which the solution is known to be a cycloid. So we have**

$$B(\mathbf{x}) = \frac{v_0 m\omega}{\alpha\sqrt{y}} \equiv \frac{b}{\sqrt{y}} \tag{9.13}$$

**The equations of motion are**

$$\ddot{x} = -\frac{b}{\sqrt{y}}\dot{y} \tag{9.14}$$

$$\ddot{y} = \frac{b}{\sqrt{y}}\dot{x} \tag{9.15}$$

**From (9.14) we have**

$$\dot{x} + b\sqrt{y} = C = 0, \tag{9.16}$$

**where we chose the initial velocity in the** $x$ **direction equal to zero. Substituting the above relation in (9.15) we obtain** $\ddot{y} = -b^2$**,** $dy/dt = b\sqrt{y_{\mathbf{max}} - y}$**, with** $y_{\mathbf{max}} = u_0^2/2b^2$**. With these we obtain**

$$\frac{dy}{dx} = -\sqrt{\frac{y_{\mathbf{max}}}{y} - 1}, \tag{9.17}$$

**the equation of the cycloid or radius** $y_{\mathbf{max}}/2$**. We stress that this analogy allows us to get the trajectories but not the dynamics of the particle. The true velocity of a particle in a spatially varying magnetic field is not necessarily related to the true dynamics of the knife edge. But the trajectories are the same. We remark that the "magnetic" terms here are different from the coefficients** $B^b_{\alpha\beta}$ **that arise in the general nonholonomic equations (2.7).**

**Conclusion:** We have shown that there is an extension of the classical optical-mechanical analogy to the nonholonomic setting that enable one to write a Hamiltonian for the trajectories of a knife edge system on the plane subject to a potential. For different potentials one obtains various classic orbital dynamic motions in the plane. The accompanying table summarizes the link between the potential of the nonholonomic system, the related index of refraction of the optical problem and the potential of the classical holonomic mechanical system that will yields the same orbit.

Table 1: Summary and examples of the non holonomic mechanical analogy for central potentials and for $V(x,y)=V(y)$

| Non-hol. potential | Index of refraction | Effective hol. potential | orbit |
|---|---|---|---|
| $V(r)$ | $n(r)=c\left(\int rdr/V^{1/2}\right)^{-1}$ | $\bar{V}(r)=-d\left(\int rdr/V^{1/2}\right)^{-2}$ | |
| -$kr^2$ | $c/r$ | $-d/r^2$ | Logarithmic spiral |
| $-k/r^2$ | $c/r^3$ | $-d/r^6$ | Lemniscate of Bernoulli |
| $V(y)$ | $c\left(\int dy/V^{1/2}\right)^{-1}$ | $-d\left(\int dy/V^{1/2}\right)^{-2}$ | |
| $C$ | $c/y$ | $-d/y^2$ | Circle |
| $ky$ | $c/\sqrt{y}$ | $-d/y$ | Cycloid |

**Nonlinear Constraints: simple example:**

**Consider an** $N$ **dimensional vector** $\mathbf{V} = (\dot{x}_1, \cdots, \dot{x}_N)$ **and an** $N$ **dimensional force** $\mathbf{F} = (f_1, \cdots, f_N)$**. The constraint is imposed by a "time dependent viscosity"** $\eta(t)$**.**

**For the velocity dependent constraint**

$$G(\mathbf{v}) = 0$$

$$\dot{\mathbf{v}} \;=\; \mathbf{F} - \eta(t)\nabla G$$

**and**

$$\dot{\mathbf{v}} \;=\; \mathbf{F} - \frac{\nabla G \cdot \mathbf{F}}{(\nabla G)^2}\nabla G$$

**guarantees that the constraint is satisfied** $dG/dt = 0$**.**

**Constant velocity constraint:**

$$G = v^2 \equiv v_0^2, \qquad (9.18)$$

$$\begin{aligned}\dot{\mathbf{v}} &= \mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{v}}{v_0^2}\mathbf{v} \\ &= \frac{\mathbf{F}(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{F}\cdot\mathbf{v})\mathbf{v}}{v_0^2} \\ &= \frac{\mathbf{v}\times(\mathbf{F}\times\mathbf{v})}{v_0^2}. \end{aligned} \tag{9.19}$$

**Using the constancy of the speed we have $\mathbf{t} = \mathbf{v}/v_0$, and**

$$\begin{aligned}\dot{\mathbf{v}} &= \frac{d\mathbf{v}}{ds}v_0 \\ &= \frac{d\mathbf{t}}{ds}v_0^2, \end{aligned} \tag{9.20}$$

**which, combined with (9.19) gives**

$$\frac{d\mathbf{t}}{ds} = \mathbf{t}\times\left(\frac{\mathbf{F}}{v_0^2}\times\mathbf{t}\right). \tag{9.21}$$

**Compare with**

$$\boxed{\frac{d\hat{\mathbf{t}}}{ds} = \hat{\mathbf{t}} \times \left(\nabla \ln(n) \times \hat{\mathbf{t}}\right)}. \tag{9.22}$$

**Given (9.19) and (9.22) we have the equivalence**

$$\mathbf{F} = -v_0^2 \nabla \ln(n), \tag{9.23}$$

**In other words, for the constant velocity constraint, the optical mechanical analogy is expressed in the equation**

$$\boxed{\frac{U(\mathbf{x})}{v_0^2} = \ln n(\mathbf{x})}. \tag{9.24}$$

**Example–constant gravity**

$$\mathbf{F} = g\hat{\mathbf{j}}$$

$$\dot{v}_y = g - g\frac{v_y^2}{v_0^2} \tag{9.25}$$

$$\dot{v}_x = -g\frac{v_y v_x}{v_0^2} \tag{9.26}$$

**Since the speed is constant, we write**

$$\mathbf{v} = v_0(\sin\theta, \cos\theta) \tag{9.27}$$

**and rewrite (9.26) as**

$$\dot{v}_x = -g\sin\theta\cos\theta. \tag{9.28}$$

**Also,**

$$\begin{aligned}\dot{v}_x &= v_0 \frac{d\sin\theta}{dy}\frac{dy}{dt} \\ &= v_0^2 \frac{d\sin\theta}{dy}\cos\theta,\end{aligned}$$

**which, combined with (9.28) gives**

$$\frac{d\sin\theta}{dy} = -\frac{g}{v_0^2}\sin\theta \tag{9.29}$$

**or**

$$\sin\theta = Ce^{-\alpha y},$$

**with $\alpha = g/v_0^2$. Now, using Snell's law**

$$n(y)\sin\theta = \text{Const}$$

**we get in fact that**

$$n(y) \propto e^{\alpha y}.$$

**In general, using (9.29)**

$$n\frac{d(1/n)}{dy} = \frac{1}{v_0^2}\frac{dV}{dy}, \tag{9.30}$$

**or**

$$-\frac{d\ln(n)}{dy} = \frac{1}{v_0^2}\frac{dV}{dy}, \tag{9.31}$$

**and**

$$\ln n(y) = -\frac{V(y)}{v_0^2} + \text{Constant} \tag{9.32}$$