Lunar perturbation of the metric associated to the averaged orbital transfer

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Riemannian metric for the unperturbed problem

Controlled Kepler equation (planar case)

$$\frac{dx}{dt} = \sum_{i=1}^{2} u_i F_i(x,l)$$
$$\frac{dl}{dt} = w_0(x,l) + g(x,l,u).$$

$x = (n, \rho, \theta)$ are the slow orbital elements of the satellite.
n is the mean motion,
ρ the eccentricity,
θ the argument of periapsis,
l is the fast angle pointing out the satellite's position on his orbit.

Averaged Hamiltonian for the energy cost

$$\overline{H} = \frac{1}{4n^{\frac{5}{3}}}\left[18n^2 p_n^2 + 5(1-\rho^2)p_\rho^2 + (5-4\rho^2)\frac{p_\theta^2}{\rho^2}\right].$$

The Riemannian metric associated

$$g = \frac{1}{9n^{\frac{1}{3}}}dn^2 + \frac{2n^{\frac{5}{3}}}{5(1-\rho^2)}d\rho^2 + \frac{2n^{\frac{5}{3}}}{5-4\rho^2}d\theta^2.$$

Model for the lunar perturbation

The perturbed dynamic can be written as

$$\frac{dx}{dt} = F_0(x,l,l') + \sum_{i=1}^{2} u_i F_i(x,l)$$

$$\frac{dl}{dt} = \widetilde{w}_0(x,l) + g(x,l,u)$$

$$\frac{dl'}{dt} = w_1(l')$$

adding l' as a fast angle pointing out the satellite's position on his orbit.

The double averaged drift vector field corresponding to the lunar perturbation is

$$\overline{F_0}(n,p,\theta) = \frac{3}{4}\frac{n'^2}{n}\sqrt{1-\rho^2}\frac{\partial}{\partial\theta}(n,p,\theta).$$

Zermelo navigation problem

A Zermelo navigation problem on a n-dimensional Riemannian manifold $(\mathscr{X}, g)$ *is a* **time minimal** *problem associated to the system*

$$\frac{dx}{dt} = X_0(x) + \sum_{i=1}^{n} u_i X_i(x)$$

where F_i *form an orthonormal frame for the metric* g *and* $|u| \leq 1$*.*

The double averaged Zermelo Hamiltonian associated to our controlled system is

$$\begin{aligned} H_{perb} &= \langle p, \overline{F}_0(x,p) \rangle + \varepsilon \sqrt{\overline{H}(x,p)} \\ &= p_\theta \left[\frac{3}{4} \frac{n'^2}{n} \sqrt{1-\rho^2} \right] + \varepsilon \sqrt{\frac{1}{4n^{\frac{5}{3}}} \left[18n^2 p_n^2 + 5(1-\rho^2) p_\rho^2 + (5-4\rho^2) \frac{p_\theta^2}{\rho^2} \right]}. \end{aligned}$$

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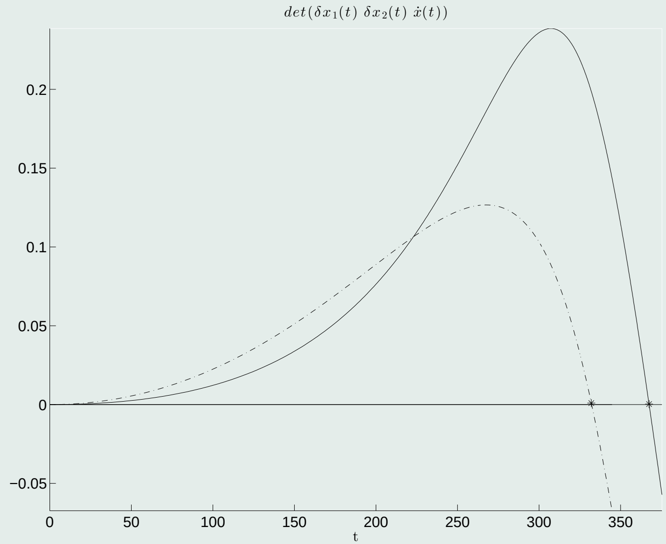
Geometric concept of conjugate point

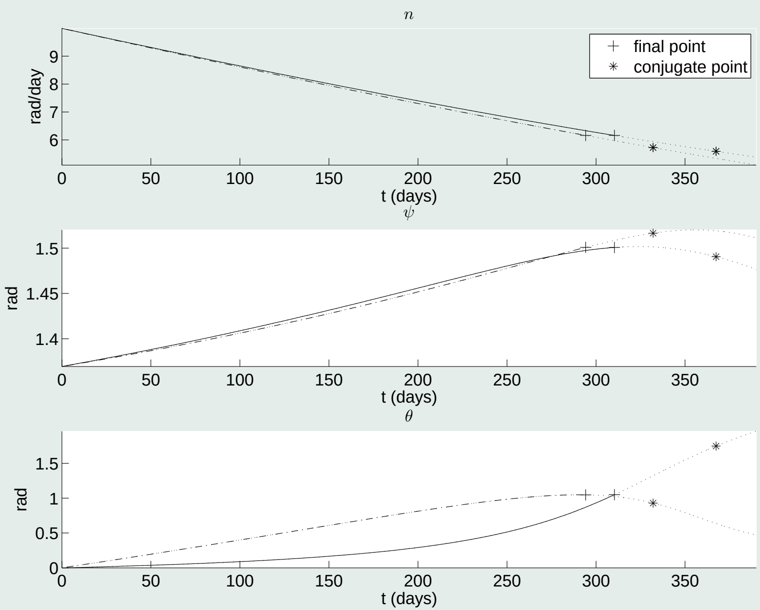
Let $\overrightarrow{H}_{pert}$ *the Hamiltonian vector field associated to* H_{pert}.

- $z = (x, p)$ *is a reference extremal, solution of* $\overrightarrow{H}_{pert}$ *on* $[0, t_f]$.
  Jacobi equation
  $$\dot{\delta}z(t) = d\overrightarrow{H}_{pert}(z(t))\delta z(t)$$
  A Jacobi field is a non trivial solution $\delta z = (\delta x, \delta p)$, *vertical at time* t *if* $\delta x(t) = 0$.
- t_c : *the first conjugate time such that the map* $p_0 \rightarrow exp_{x_0}(t, p_0) = x(t, x_0, p_0)$ *is not an immersion at* $t = t_c$.
- Rank test condition to compute t_c for a given trajectory $x(t, x_0, p_0)$

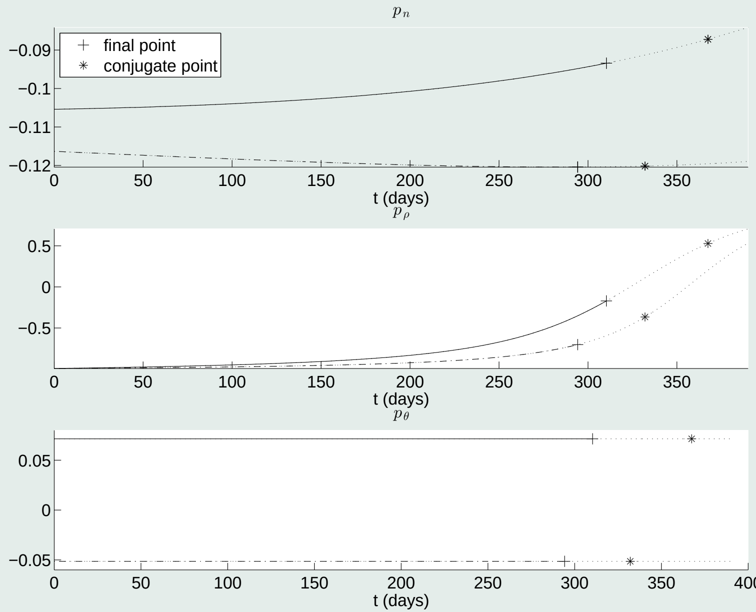
  $$det[\delta x_1(t_c), \ldots, \delta x_{n-1}(t_c), \dot{x}(t_c)] = 0.$$



Time evolution of trajectories



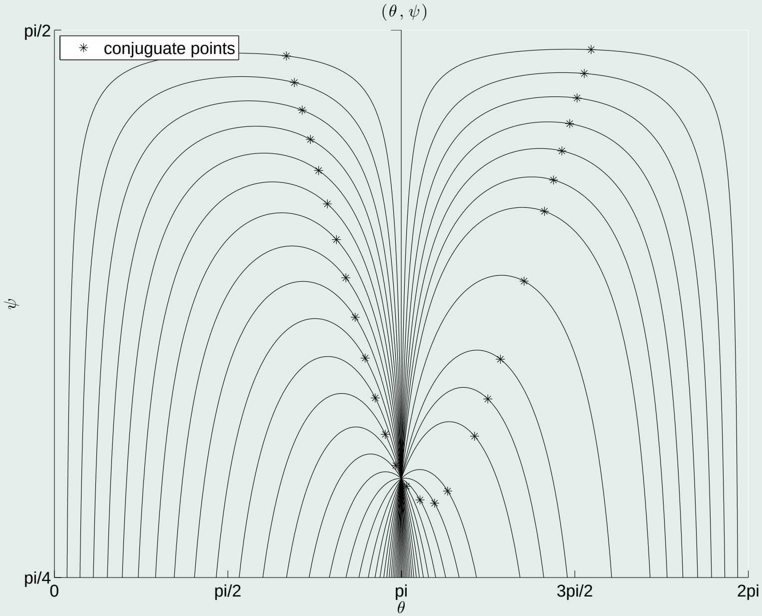
Evolution of the state vector
$(n(t), \psi(t) = \frac{\pi}{2} - asin(\rho(t)), \theta(t))$.



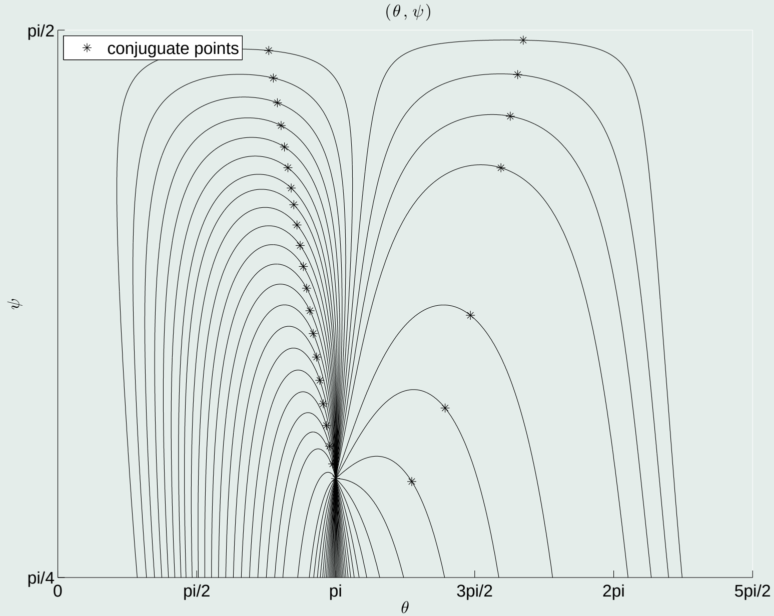
Evolution of the adjoint vector
$(p_n(t), p_\rho(t), p_\theta(t))$.

solide line : unperturbed case,
dash-dot line : perturbed case.

Trajectories in (θ, ψ) coordinates



Trajectories from the unperturbed case.



Trajectories from the perturbed case.