From dynamics to geometry:

from control systems and dynamic pairs to canonical connection and curvature

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Part I: Main definition and motivation

- Dynamic pairs (1-regular control systems)
- Special classes:
  - Second order differential equations
  - Lagrangian systems
  - Mechanical dynamical systems
  - Fully actuated control systems

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Part II: Dynamic pairs and their geometry

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- Normal vector fields
- Canonical splitting
- Jacobi endomorphism (curvature)
- Canonical connection

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Part III: Lagrangian systems

The talk is based on a joint work with W. Kryński
(J. Geometric Mechanics 2013)

The usual way of studying "classical physical systems" is to go:

From geometry to dynamics

One of the "mottos" of this talk:

What if go the other way?

This way is suggested by geometric control theory.

Part I:

Main definition and motivations

Dynamic pair

The main objects to discuss in this lecture will be dynamic pairs.

Def. Dynamic pair on a manifold M is a pair $(X, \mathcal{V})$, where:

- X - a smooth vector field on M,
- $\mathcal{V}$ - a smooth distribution on M (possibly nonintegrable).

Such pair is called regular if, for $x \in M$,

- $\dim \mathcal{V}(x) = n$ and $\dim M = 2n$,
- $\mathcal{V} + [X, \mathcal{V}] = TM$,
- $X(x) \neq 0$.

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Alternatively, $(X, \mathcal{V})$ is called regular if, for $x \in M$,

- $\dim \mathcal{V}(x) = n$ and $\dim M = 2n + 1$,
- $\mathcal{X} + \mathcal{V} + [X, \mathcal{V}] = TM$,

where $\mathcal{X}$ is the 1-dimensional distribution spanned by X.

Regular control systems

Def A control system

$$\Sigma : \quad \dot{x} = X(x) + \sum_{j=1,\dots,n} u_j Y_j(x), \quad x \in M,$$

is regular (more exactly, 1-regular) if either the vector fields

$$Y_1, \dots, Y_n, \ [X, Y_1], \dots, [X, Y_n], \qquad (R)$$

or

$$X, Y_1, \dots, Y_n, \ [X, Y_1], \dots, [X, Y_n] \qquad (R')$$

are pointwise linearly independent and span TM.

Example: fully actuated mechanical systems are 1-regular.

Defining X - the drift of the above system,
$\mathcal{V} = \mathrm{span}\,\{Y_1, \dots, Y_n\}$ and assuming regularity gives a regular dynamic pair $(X, \mathcal{V})$.

k-regular control systems (not discussed in this lecture) require to take up to k Lie brackets with X. They can be analyzed by similar methods (see the joint paper with W. Kryński).

Second order dynamical system

By second order dynamical system DS we mean

$$\ddot{q} = F(t, q, \dot{q}), \quad q \in \mathbb{R}^n.$$

Denoting $v = \dot{q}$, it can be written as:

$$\dot{q} = v, \quad \dot{y} = F(t, q, v), \qquad (DS)$$

where $q = (q^1, \ldots, q^n) \in \mathbb{R}^n$, $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$, and $F = (F^1, \ldots, F^n)$. Then the dynamics is represented by the vector field on $\mathbb{R}^n \times \mathbb{R}^n$

$$X = v^i \partial_{q^i} + F^i(q, v) \partial_{v^i}.$$

The distribution $\mathcal{V}$ is given by $\mathcal{V} = \mathrm{span}\{\partial_{v^1}, \ldots, \partial_{v^n}\}$.
The pair $(X, \mathcal{V})$ is a regular dynamic pair.

Lagrangian systems

Consider a system described by:

- a configuration manifold Q and "phase manifold" $M = TQ$,
- a regular Lagrange function $L : TQ \to \mathbb{R}$.

The dynamics is described by Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \qquad (EL)$$

They can be brought to a system of first order equations

$$\dot{q} = v, \quad \dot{v} = F(q, v),$$

where, putting $g_{ij} = \partial^2 L / \partial v^i \partial v^j$, $(g^{ij}) = (g_{ij})^{-1}$, we define

$$F^i = \frac{1}{2} g^{ij} \left(\frac{\partial^2 L}{\partial v^j \partial q^k} v^k - \frac{\partial L}{\partial q^i} \right).$$

Equations (EL) define a vector field X on $M = TQ$ which, together with the distribution $\mathcal{V} = \mathrm{span}\{\partial_{v^1}, \ldots, \partial_{v^n}\}$, form a dynamic pair.

Classical schemes

Given the geometry of a system

(e.g., mass-inertia metric or Lagrange function)

$\Downarrow$ deduce

Dynamics

Possibly, add additional forces to dynamic equations by hand.

In Geometric Mechanics

Geometry of system given by:

- Q - configuration manifold
- g - Riemann metric on Q given by masses and inertia

$$\Downarrow \quad \text{deduce}$$

Dynamics:

$$\frac{D\dot{q}}{dt} = 0,$$

where D/dt denotes the covariant derivative corresponding to the Levi-Civita connection of g.
More generally:

$$g\left(\frac{D\dot{q}}{dt}, \ \cdot \ \right) = \mathcal{F}(q, \dot{q}),$$

where $\mathcal{F}$ denotes external forces.

In Lagrange formalism

Geometry given by Lagrange function $L : TQ \to \mathbb{R}$

$$\Downarrow \quad \text{deduce}$$

Dynamics given by Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

In all these cases the "physical truth" is given by dynamical equations consistent with experiments.
The "geometry" is postulated by us, for convenience.

In the real world we do not see the metric, we see the movement (dynamics) and the symmetry of physical laws (with respect to Euclidean or Poincaré groups). The symmetry suggests the metric.

Question: Which part of the geometry follows from the dynamics?

What is "dynamics"?

My proposal: the dynamics consists of:

M - phase space

X - a vector fields on M

$\mathcal{V} \subset TM$ - a distribution modeling possible external forces (or perturbations, disturbances, admissible variations).

Classically:
$M = TQ$ - tangent bundle to configuration manifold, $(q, v) \in TQ$,
$\mathcal{V} = \mathrm{span}\{\partial_{v^1}, \ldots, \partial_{v^n}\}$ - vertical distribution of the tangent bundle $M = TQ \to Q$,
X - a spray on TQ, i.e. a vector field of the form

$$X = v^i \partial_{q^i} + S^i(q, v)\partial_{v^i},$$

in local coordinates, where $S^i(q, \lambda v) = \lambda^2 S^i(q, v)$, for $\lambda > 0$.

Our "dynamics" is more general since $\mathcal{V}$ is not integrable.

Let:

- $M = TQ$, $x = (q, v) \in TQ$, $\dim Q = n$,
- $\mathcal{V} = \mathrm{span}\,\{\partial_{v^1}, \ldots, \partial_{v^n}\}$ - vertical distribution of the tangent bundle $M = TQ \to Q$,
- X - a semispray on TQ, i.e. a vector field of the form

$$X = \sum v^i \partial_{q^i} + S^i(q, v)\partial_{v^i}.$$

Then

$$[X, \partial_{v^j}] = -\partial_{q^j} - \frac{\partial S^i}{\partial v^i}\partial_{v^i}$$

and

$$\mathrm{span}\,\{\partial_{v^1}, \ldots, \partial_{v^n}, [X, \partial_{v^1}], \ldots, [X, \partial_{v^n}]\} = TM$$

Thus $\dim \mathcal{V}(x) = n$, $\dim \mathrm{span}\,\{\mathcal{V}, [X, \mathcal{V}]\} = 2n$, i.e.,

$$\mathcal{V} + [X, \mathcal{V}] = TM.$$

Part II:

Geometry of (regular) dynamic pairs

Dynamic pairs (recall)

Def. Dynamic pair on a manifold M is a pair $(X, \mathcal{V})$, where:

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Such pair is called regular if, for $x \in M$,

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We will use the first definition (both versions lead to similar constructions).

Normal bases of $\mathcal{V}$

We will use special local bases (frames) of $\mathcal{V}$

$$\mathcal{V}(x) = \mathrm{span}\,\{V_1(x), \ldots, V_n(x)\}.$$

Def. A vector field $V \in \mathcal{V}$ is normal if

$$[X, [X, V]] \in \mathcal{V}.$$

Def. A basis $V_1, \ldots, V_n$ of $\mathcal{V}$ is normal if $\exists$ functions K_j^i s.t.

$$[X, [X, V_j]] \;=\; K_j^i \, V_i.$$

Basic Lemma

- Normal bases exist, locally.
- Two normal bases $\bar{V} = (V_1, \ldots, V_n)$ and $\bar{V}' = (V_1', \ldots, V_n')$ are related by an invertible matrix of functions $G = (G_j^i)$,

$$V_j = G_j^i V_i', \quad \text{where} \quad X(G_j^i) = 0.$$

- The matrix $K = (K_i^j)$ defines an endomorphism of the vector bundle $\mathcal{V}$, i.e. it defines linear operators

$$K(x) : \mathcal{V}(x) \to \mathcal{V}(x)$$

Horizontal distribution, canonical splitting

A local normal basis (frame) $(V_1, \ldots, V_n)$ of $\mathcal{V}$ defines another canonical distribution

$$\mathcal{H} = \mathrm{span}\,\{[X, V_1], \ldots, [X, V_n]\}.$$

Def. $\mathcal{H}$ is called horizontal distribution of the dynamic pair.

Because of regularity of the dynamic pair $(X, \mathcal{V})$ we have the splitting

$$TM = \mathcal{V} \oplus \mathcal{H},$$

called canonical splitting. The corresponding pointwise projections are denoted

$$\pi_{\mathcal{V}} : TM \to \mathcal{V}, \quad \pi_{\mathcal{H}} : TM \to \mathcal{H}.$$

Canonical isomorphism A and Jacobi curvature

Def. The canonical isomorphism $A : \mathcal{V} \to \mathcal{H}$ of vector bundles is defined by

$$A\,V = \pi_{\mathcal{H}}[X, V], \quad V \in \mathcal{V}.$$

If V is normal, then $[X, V] \in \mathcal{V}$ and $AV = [X, V]$.

Def. Jacobi curvature of the pair $(X, \mathcal{V})$ is the map $K : \mathcal{V} \to \mathcal{V}$ defined by

$$K = -BA,$$

where

$$B : \mathcal{H} \to \mathcal{V}, \quad B\,H = \pi_{\mathcal{V}}[X, H], \;\; H \in \mathcal{H}.$$

In a normal basis the matrix of K is defined by the equations

$$[X, [X, V_j]] = -K_j^i V_i$$

(we use the summation convention).

Explicit formulae

In any basis $(V_1, \ldots, V_n)$ of $\mathcal{V}$ we can compute the horizontal distribution $\mathcal{H}$, the canonical isomorphism $A : \mathcal{V} \to \mathcal{H}$ and the Jacobi curvature K as follows.

Introduce $n \times n$ matrices of functions H_0 and H_1 by:

$$[X, [X, V_j]] = (H_0)^i_j \, V_i + (H_1)^i_j \, [X, V_i]$$

(we use the regularity assumption). Then

$$H_j = A\, V_j = [X, V_j] - \frac{1}{2}(H_1)^i_j V_j,$$

$$\mathcal{H} = \mathrm{span}\,\{H_1, \ldots, H_n\},$$

$$K = -H_0 + \frac{1}{2} X(H_1) - \frac{1}{4} H_1^2.$$

Classical case of SODE

For dynamic pairs given by systems of second order diff. equations (SODE) we have

$$X = v^i \partial_{q^i} + F^i(q, v)\partial_{v^i}$$

and the vertical distribution is $\mathcal{V} = \mathrm{span}\,\{\partial_{v^1}, \dots, \partial_{v^n}\}$.
A simple computation gives

$$H_0 = F_q - X(F_v), \quad H_1 = -F_v,$$

$$F_q := \frac{\partial F}{\partial q}, \quad F_v := \frac{\partial F}{\partial v}.$$

Our formulae for $\mathcal{H} = \mathrm{span}\,\{H_1, \dots, H_n\}$ and K give the classical:

$$H_j = \partial_{q^j} + \frac{1}{2} F^i_{v^j} \partial_{v^i}, \quad F^i_{v^j} := \frac{\partial F^i}{\partial v_j}.$$

$$K = -F_q + \frac{1}{2} X(F_v) - \frac{1}{4} F_v^2.$$

Remark 1

The canonical isomorphism $A : \mathcal{V} \to \mathcal{H}$ seems not present in the classical setting but it can be deduced by the so called "almost tangent structure" operator $J : TQ \to TQ$ from the relation

$$A^{-1} = J\tau_*|_{\mathcal{H}},$$

where $\tau : TQ \to Q$ is the canonical projection.

Remark 2: almost complex structure

A regular dynamic pair $(X, \mathcal{V})$ on M defines a canonical almost complex structure J in the tangent bundle TM. The endomorphism $J : TM \to TM$ is defined, in the decomposition $TM = \mathcal{V} \oplus \mathcal{H}$,by the operator

$$J = \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}.$$

Then $J^2 = -I$ and each tangent space T_xM is endowed with the complex structure defined by $J(x)$, where the operator $J(x) : T_xM \to T_xM$ represents multiplication by $i = \sqrt{-1}$. In general, this structure is not integrable.

Canonical connection

A regular dynamical pair $(X, \mathcal{V})$ defines a class of natural linear connections $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which, by definition, satisfy:

- They preserve the splitting $TM = \mathcal{V} \oplus \mathcal{H}$, i.e.,

$$\nabla_Y : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}), \quad , \nabla_Y : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H}), \ \forall \ Y \in \Gamma(TM).$$

- They commute with the canonical isomorphism $A : \mathcal{V} \to \mathcal{H}$:

$$\nabla_Y(AZ) = A\nabla_Y Z, \quad \forall \ Y \in \Gamma(TM), \ \forall \ Z \in \Gamma(\mathcal{V}).$$

There is one special connection in this class:

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There is one special connection in this class:

THM There is a unique connection satisfying the above conditions and

$$T(V, H) = 0 \quad \forall \; V \in \Gamma(\mathcal{V}), \;\; \forall \; H \in \Gamma(\mathcal{H}),$$

where T is the torsion tensor of ∇,

$$T(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z].$$

We call such ∇ canonical connection assigned to the pair $(X, \mathcal{V})$.

Explicit formulae for canonical connection

The Christoffel coefficients of the canonical connection, in the basis $V_1, \ldots, V_n \in \mathcal{V}$, $H_1, \ldots, H_n \in \mathcal{H}$, are defined via the commutation relations

$$[H_i, V_j] = \Gamma_{ij}^k V_k - \widetilde{\Gamma}_{ij}^k H_k.$$

Then

$$\nabla_{H_i} V_j = \Gamma_{ij}^k V_k, \qquad \nabla_{V_i} V_j = \widetilde{\Gamma}_{ij}^k V_k,$$

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Using the ad operator, $\mathrm{ad}_Y Z = [Y, Z]$), the basis independent formulae for ∇, with $V, V' \in \mathcal{V}$, $H, H' \in \mathcal{H}$, are:

$$\nabla_H V = \pi_{\mathcal{V}} \mathrm{ad}_H V, \qquad \nabla_V V' = A^{-1} \pi_{\mathcal{H}} \mathrm{ad}_V (AV'),$$

$$\nabla_H H' = A \pi_{\mathcal{V}} \mathrm{ad}_H (A^{-1} H'), \quad \nabla_V H = \pi_{\mathcal{H}} \mathrm{ad}_V H.$$

Remarks

- Fixing an arbitrary bilinear tensor map $N : \mathcal{V} \times \mathcal{H} \to TM$ and imposing the condition

$$T(V, H) = N(V, H), \quad \forall\ V \in \Gamma(\mathcal{V}), \ \ \forall\ H \in \Gamma(\mathcal{H})$$

on a natural linear connection ∇ also makes this connection unique.

- If the dynamic pair $(X, \mathcal{V})$ is defined by a system of 2-nd order equations

$$\dot{q} = v, \quad \dot{v} = F(q, v)$$

and F has the property

$$F(q, \lambda v) = \lambda^2 F(q, v)$$

(i.e., it defines a spray), then the canonical connection satisfies

$$\nabla_X X = 0,$$

that is, the trajectories of X are geodesics of ∇.
Without this property this is not true (however, it is always true for an analogous construction with the second regularity condition).

Part III:

Geometry of Lagrangian systems (Lagrangian Geometry)

This is an extension of Finsler Geometry
(the work of J. Kern (1975),
R. Miron and Romanian school (80-ties and 90-ties) ...)

Lagrangian systems (again)

Let $M = TQ$ and consider a regular Lagrange function $L : TQ \to \mathbb{R}$.

Choose a coordinate system $q^1, \dots, q^n$, $v^1, \dots, v^n$ and assume, in addition, that the metric defined by on the fibers of $TQ \to Q$ by the coefficients

$$g_{ij}(q, v) = \frac{\partial^2 L}{\partial v^i \partial v^j},$$

has constant signature. Then (Q, L) is called Lagrange space.

The dynamics is described by Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

can be brought to a system of first order equations

$$\dot{q} = v, \quad \dot{v} = F(q, v),$$

where, putting $(g^{ij}) = (g_{ij})^{-1}$, we define

$$F^i = \frac{1}{2} g^{ij} \left(\frac{\partial^2 L}{\partial v^j \partial q^k} v^k - \frac{\partial L}{\partial q^i} \right).$$

Thus, the Euler-Lagrange equations define a vector field X_L on $M = TQ$ which, together with the distribution $\mathcal{V} = \mathrm{span}\{\partial_{v^1}, \ldots, \partial_{v^n}\}$, form a regular dynamic pair.

• Our horizontal distribution $\mathcal{H}$ corresponding to the dynamic pair $(X_L, \mathcal{V})$ coincides with the one known (and constructed in a different way) in Lagrangian Geometry.

• Similarly, the Jacobi curvature also appears in Lagrange geometry.

• A canonical metric is defined on the tangent bundle TM (where $M = TQ$) by taking, in the basis $V_i = \partial_{v^i}$, $H_i = AH_i$,

$$g(V_i, V_j) = g(H_i, H_j) = g_{ij}, \quad g(V_i, H_j) = 0.$$

• A canonical linear connection is constructed on TM which preserves the metric.
However, this connection does not coincide with ours, applied to this special case. In particular, the equality $\nabla_X X = 0$ is not true for this connection!

Example: a charged particle in EM field

Consider the equations of charged particle moving in the electromagnetic field

$$\ddot{q} = a(\vec{E}(t,q) + \dot{q} \times \vec{B}(t,q)) := F(t,q,\dot{q}) \qquad (Lo)$$

where $a = e/m$. Write $\dot{x} \times \vec{B} = -\hat{B}\dot{x}$ where $\hat{B} = (\hat{B}^i_j)$ is the antisymmetric matrix

$$\begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}.$$

Prop. 1 If the field $(\vec{E}, \vec{B})$ satisfies the Maxwell equations $\partial \vec{B}/\partial t = -\nabla \times \vec{E}$ and $\mathrm{div}\, \vec{B} = 0$, then the curvature matrix is symmetric and equal to

$$K = -a(\vec{E}_x)_{\mathrm{sym}} - a((\hat{B}\dot{x})_x)_{\mathrm{sym}} - \frac{1}{4}a^2\hat{B}^2,$$

where $A_{\mathrm{sym}} = (A + A^T)/2$ denotes the symmetric part of matrix A.

Prop. 2 Assume that $\vec{E}$ and $\vec{B}$ satisfy the Maxwell equations

$$\mathrm{div}\vec{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}$$

in SI units, where ϵ_0 and μ_0 are the electric and magnetic constants, respectively, ρ is the density of charge and $\vec{J}$ is the field of electric current. Then, denoting $E_t = \partial\vec{E}/\partial t$, we have

$$\mathrm{tr}\, K = -\frac{a}{\epsilon_0}\rho + a\mu_0\langle \dot{x}, \vec{J} + \epsilon_0\vec{E}_t\rangle + \frac{a^2}{2}|\vec{B}|^2.$$

The matrix K and its trace make possible to estimate the conjugate points along a given trajectory of the particle.

Concluding remarks and questions

- Dynamic pairs $(X, \mathcal{V})$ describe more general objects then those obtained from Lagrange geometry or geometry of second order differential equations.

- They define geometric structures analogous to classical (but not all). Some of them differ from classical and have better properties.

- They define the geometry of the class of 1-regular control systems, in particular, fully actuated control systems (e.g., mechanical robots).

Question. Do there exist physical systems where the "vertical distribution" is not integrable, in particular the phase space is not the tangent bundle to a configuration manifold (configuration manifold can not be defined)?