First and Second order Sensitivity Relations in Optimal Control

Hélène Frankowska

CNRS and UNIVERSITÉ PIERRE et MARIE CURIE

In collaboration with P. CANNARSA and T. SCARINCI

Nonholonomic mechanics and optimal control

IHP, November 28, 2014

Outline of the talk

Mayer's Optimal Control Problem

Consider the minimization problem

$$V(t_0, x_0) := \inf \left\{ \phi(x(T)) \ : \ x(\cdot) \in S_{[t_0,T]}(x_0) \right\} \qquad \mathcal{P}(t_0, x_0)$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ and $S_{[t_0,T]}(x_0)$ is the set of all absolutely continuous solutions of the **control system**

$$\begin{cases} x' = f(x, u(t)), \quad u(t) \in U \quad \text{a.e. in } [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ and U is a complete separable metric space. $V : (-\infty, T] \times \mathbb{R}^n \to \mathbb{R}$ is called the **value function**.

Standard Hypothesis: ϕ is locally Lipschitz and

$$\begin{cases} (i) & f(x, U) \text{ is compact for each } x \in \mathbb{R}^n \\ (ii) & f(\cdot, u) \text{ is locally Lipschitz uniformly in } u \in U \\ (iii) & \exists\, \gamma > 0 \text{ so that } \max\{|f(x,u)| : u \in U\} \leq \gamma(1 + |x|)\ \forall x \in \mathbb{R}^n \end{cases}$$

Mayer's Optimal Control Problem

Consider the minimization problem

$$V(t_0, x_0) := \inf \left\{ \phi(x(T)) \ : \ x(\cdot) \in S_{[t_0,T]}(x_0) \right\} \qquad \mathcal{P}(t_0, x_0)$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ and $S_{[t_0,T]}(x_0)$ is the set of all absolutely continuous solutions of the **control system**

$$\begin{cases} x' = f(x, u(t)), \quad u(t) \in U \quad \text{a.e. in } [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ and U is a complete separable metric space. $V : (-\infty, T] \times \mathbb{R}^n \to \mathbb{R}$ is called the **value function**.

Standard Hypothesis: ϕ is locally Lipschitz and

$$\begin{cases} (i) & f(x, U) \text{ is compact for each } x \in \mathbb{R}^n \\ (ii) & f(\cdot, u) \text{ is locally Lipschitz uniformly in } u \in U \\ (iii) & \exists\, \gamma > 0 \text{ so that } \max\{|f(x,u)| : u \in U\} \leq \gamma(1 + |x|)\ \forall x \in \mathbb{R}^n \end{cases}$$

Generalized Differentials

Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$. For any $x \in \Omega$, the sets

$$\partial^- f(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|\, y - x \,|} \geq 0 \right\}$$

$$\partial^+ f(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|\, y - x \,|} \leq 0 \right\}$$

are the **(Fréchet) subdifferential** and **superdifferential** of f at x, respectively.

$p \in \mathbb{R}^n$ is a **proximal subgradient** of f at $x \in \Omega$ if $\exists\ c,\ \rho \geq 0$

$$f(y) - f(x) - \langle p, y - x \rangle \geq -c|y - x|^2 \quad \forall y \in B(x, \rho).$$

The set of all proximal subgradients of f at x is denoted by $\partial^{-,pr} f(x)$.

Generalized Differentials

Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$. For any $x \in \Omega$, the sets

$$\partial^- f(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|\, y - x \,|} \geq 0 \right\}$$

$$\partial^+ f(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|\, y - x \,|} \leq 0 \right\}$$

are the **(Fréchet) subdifferential** and **superdifferential** of f at x, respectively.

$p \in \mathbb{R}^n$ is a **proximal subgradient** of f at $x \in \Omega$ if $\exists\ c,\ \rho \geq 0$

$$f(y) - f(x) - \langle p, y - x \rangle \geq -c|y - x|^2 \quad \forall y \in B(x, \rho).$$

The set of all proximal subgradients of f at x is denoted by $\partial^{-,pr} f(x)$.

HJB and Characteristics

Define the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$H(x,p) = \sup_{u \in U} \langle f(x,u), p \rangle \qquad \forall\ (x,p) \in \mathbb{R}^n \times \mathbb{R}^n$$

V is the unique solution, in a suitable sense, of the Hamilton-Jacobi equation

$$\begin{cases} -v_t(t,x) + H(x, -v_x(t,x)) = 0 & \text{in } (-\infty, T) \times \mathbb{R}^n \\ v(T,x) = \phi(x) & x \in \mathbb{R}^n \end{cases}$$

Characteristic system : $\quad p(t) = -v_x(t, x(t))$

$$\begin{cases} \dot{x}(t) = \nabla_p H(x(t), p(t)) & x(T) = x_T \\ -\dot{p}(t) = \nabla_x H(x(t), p(t)) & -p(T) = \nabla \phi(x_T) \end{cases}$$

where $\nabla_x H$ is the gradient of $H(\cdot, p)$, similarly for $\nabla_p H$ whenever $p(\cdot) \neq 0$

HJB and Characteristics

Define the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$H(x,p) = \sup_{u \in U} \langle f(x,u), p \rangle \qquad \forall\, (x,p) \in \mathbb{R}^n \times \mathbb{R}^n$$

V is the unique solution, in a suitable sense, of the Hamilton-Jacobi equation

$$\begin{cases} -v_t(t,x) + H(x, -v_x(t,x)) = 0 & \text{in } (-\infty, T) \times \mathbb{R}^n \\ v(T,x) = \phi(x) & x \in \mathbb{R}^n \end{cases}$$

Characteristic system : $p(t) \neq -v_x(t, x(t))$, in general,

$$\begin{cases} \dot{x}(t) \in \partial_p^- H(x(t), p(t)) & x(T) = x_T \\ -\dot{p}(t) \in \partial_x^- H(x(t), p(t)) & -p(T) \in \partial\phi(x_T) \end{cases}$$

where $\partial_x^- H$ is the subdifferential of $H(\cdot, p)$, similarly for $\partial_p^- H$ and $\partial\phi$ is the generalized gradient of ϕ

Sensitivity Relations for Smooth Control Systems

Let $\bar{x}(\cdot)$ be an optimal solution for $\mathcal{P}(t_0, x_0)$.

If f and ϕ are sufficiently smooth and V is differentiable, then the adjoint state $p(\cdot)$ (of the **maximum principle**) satisfies the partial sensitivity relation

$$-p(t) = V_x(t, \bar{x}(t)) \quad \forall t \in [t_0, T]$$

and the full sensitivity relation

$$(H(\bar{x}(t), p(t)), -p(t)) = \nabla V(t, \bar{x}(t)) \quad \forall t \in [t_0, T)$$

This leads to necessary and sufficient conditions for optimality.

Sensitivity Relations for Smooth Control Systems

Let $\bar{x}(\cdot)$ be an optimal solution for $\mathcal{P}(t_0, x_0)$.

If ϕ is locally Lipschitz, then by Clarke, Vinter 1987 there exists an adjoint state $p(\cdot)$ (co-state from the maximum principle) satisfying

$$-p(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{a.e. in } [t_0, T]$$

and by Vinter 1988 there exists an adjoint state $q(\cdot)$ satisfying

$$(H(\overline{x}(t), q(t)), -q(t)) \in \partial V(t, \overline{x}(t)) \text{ for all } t \in (t_0, T)$$

This relations do not imply sufficient conditions for optimality. They also hold true in the state constrained case.

Sensitivity Relations for Smooth Control Systems

Let $\bar{x}(\cdot)$ be an optimal solution for $\mathcal{P}(t_0, x_0)$.

If $f(\cdot, u)$ and ϕ are differentiable, then the adjoint state $p(\cdot)$ satisfies

$$-p(t) \in \partial_x^+ V(t, \bar{x}(t)) \quad \forall t \in [t_0, T]$$

and

$$(H(\bar{x}(t), p(t)), -p(t)) \in \partial^+ V(t, \bar{x}(t)) \qquad \text{a.e. in } [t_0, T]$$

Subbotina 1989, Cannarsa and HF 1990.

This leads to necessary and sufficient conditions for optimality.

Not investigated **yet** when state constraints are present.

Semiconcave and Semiconvex Functions

Let $\Omega \subset \mathbb{R}^n$, $c \geq 0$, $f : \Omega \to \mathbb{R}$ is $c-$**semiconcave** if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)c|x-y|^2$$

for all $x, y \in \Omega$ such that $[x, y] \subset \Omega$ and $\lambda \in [0, 1]$.
f is called $c-$**semiconvex** on Ω if $-f$ is $c-$semiconcave on Ω.

If f is sufficiently smooth in x and ϕ is C^2, then the value function is locally semiconcave.

Hence it has directional derivatives and if the subdifferential of V is nonempty at some (t, x), then V is differentiable at this point. V is then the unique locally Lipschitz solution of HJB equation in the classical sense.

Semiconcave and Semiconvex Functions

Let $\Omega \subset \mathbb{R}^n$, $c \geq 0$, $f : \Omega \to \mathbb{R}$ is $c-$**semiconcave** if

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)c|x-y|^2$$

for all $x, y \in \Omega$ such that $[x, y] \subset \Omega$ and $\lambda \in [0, 1]$.
f is called $c-$**semiconvex** on Ω if $-f$ is $c-$semiconcave on Ω.

If f is sufficiently smooth in x and ϕ is C^2, then the value function is locally semiconcave.

Hence it has directional derivatives and if the subdifferential of V is nonempty at some (t, x), then V is differentiable at this point. V is then the unique locally Lipschitz solution of HJB equation in the classical sense.

Maximum Principle

Theorem

Assume $\forall\ r > 0\ \exists\ c \geq 0$ such that $\forall\ p \in S^{n-1}$, $x \mapsto H(\cdot, p)$ is $c-$semiconvex on $B(0,r)$.
If $\bar{x}(\cdot)$ is optimal for $\mathcal{P}(t_0, x_0)$, then there exists an arc $p : [t_0, T] \to \mathbb{R}^n$ which, together with $\bar{x}(\cdot)$, satisfies

$$\begin{cases} \dot{x}(s) & \in & \partial_p^- H(x(s), p(s)), \\ -\dot{p}(s) & \in & \partial_x^- H(x(s), p(s)), \end{cases} \quad \text{for a.e.} \quad s \in [t_0, T]$$

and $-p(T) \in \partial\phi(x(T))$.

Clarke, Vinter show it with $\partial H(x,p)$ which is contained in $\partial_p^- H(x,p) \times \partial_p^- H(x,p)$ because of semiconvexity of Hamiltonian.

Sufficient Conditions for Optimality

Theorem

Let $x(\cdot) \in S_{[t_0,T]}(x_0)$. If, for almost every $t \in [t_0, T]$, $\exists\, p(t) \in \mathbb{R}^n$

$$\langle p(t), \dot{x}(t)\rangle = H(x(t), p(t))$$
$$(H(x(t), p(t)), -p(t)) \in \partial^+ V(t, x(t))$$

then x is optimal for $\mathcal{P}(t_0, x_0)$.

Regularity Assumptions: $\forall\, r > 0,\ \exists\, c \geq 0,\ \forall\, p \in S^{n-1}$

$$\begin{cases} (i)\ x \mapsto H(x,p) \text{ is c-semiconvex on } B(0,r) \\ (ii)\ \nabla_p H(x,p) \text{ exists and is c-Lipschitz continuous in } x \text{ on } B(0,r) \end{cases}$$

If ϕ is locally semiconcave, then V is also locally semiconcave
Cannarsa and Wolenski, 2011

Sensitivity Relations Involving Superdifferentials

Theorem

Let $\overline{x}(\cdot)$ be optimal for $\mathcal{P}(t_0, x_0)$ and consider any arc $\overline{p}(\cdot)$ such that $(\overline{x}, \overline{p})$ solves the system

$$\begin{cases} -\dot{p}(t) & \in & \partial_x^- H(x(t), p(t)) \\ \dot{x}(t) & \in & \partial_p^- H(x(t), p(t)) \end{cases} \qquad -p(T) \in \partial^+ \phi(\overline{x}(T))$$

Then $\overline{p}(\cdot)$ satisfies the full sensitivity relation

$$(H(\overline{x}(t), \overline{p}(t)), -\overline{p}(t)) \in \partial^+ V(t, \overline{x}(t)) \text{ for all } t \in (t_0, T)$$

and the partial sensitivity relation

$$-\overline{p}(t) \in \partial_x^+ V(t, \overline{x}(t)) \text{ for all } t \in [t_0, T]$$

Sensitivity Relations Involving Subdifferentials

Theorem

Assume $\partial_x^- V(t_0, x_0) \neq \emptyset$. *Let* $\overline{x}(\cdot)$ *be optimal for* $\mathcal{P}(t_0, x_0)$ *and consider* *any* *arc* $\overline{p}(\cdot)$ *such that* $(\overline{x}, \overline{p})$ *solves the system*

$$\begin{cases} \dot{x}(s) & \in & \partial_p^- H(x(s), p(s)), & x(t_0) & = & x_0 \\ -\dot{p}(s) & \in & \partial_x^- H(x(s), p(s)), & -p(t_0) & \in & \partial_x^- V(t_0, x_0) \end{cases}$$

Then $-\overline{p}(t) \in \partial_x^- V(t, \overline{x}(t))$ *for all* $t \in [t_0, T]$.

Furthermore, if $\partial^+ \phi(\overline{x}(T)) \neq \emptyset$, *then for all* $t \in [t_0, T]$, $V(t, \cdot)$ *is differentiable at* $\overline{x}(t)$ *and* $\nabla_x V(t, \overline{x}(t)) = -\overline{p}(t)$.
If ϕ *is also locally semiconcave, then* $V(\cdot, \cdot)$ *is differentiable at* $(t, \overline{x}(t))$ *and* $\nabla V(t, \overline{x}(t)) = (H(\overline{x}(t), \overline{p}(t)), -\overline{p}(t))\ \forall\ t \in [t_0, T)$.

Second Order Superjets and Subjets

$S(n)$ is the set of symmetric $n \times n$ matrices.
Let $f : \mathbb{R}^n \to [-\infty, +\infty]$ and $x \in dom(f)$.

$(q, Q) \in \mathbb{R}^n \times S(n)$ is a **superjet** of f at x if $\exists\, \delta > 0$, $\forall\, y \in B(x, \delta)$

$$f(y) \leq f(x) + \langle q, y - x\rangle + \frac{1}{2}\langle Q(y - x), y - x\rangle + o(|\, y - x\,|^2)$$

The set of all the superjets of f at x is denoted by $J^{2,+}f(x)$.

$(q, Q) \in \mathbb{R}^n \times S(n)$ is a **subjet** of f at x if $\exists\, \delta > 0$, $\forall\, y \in B(x, \delta)$

$$f(y) \geq f(x) + \langle q, y - x\rangle + \frac{1}{2}\langle Q(y - x), y - x\rangle + o(|\, y - x\,|^2).$$

The set of all the subjets of f at x is denoted by $J^{2,-}f(x)$.

Properties of Superjets

Proposition

Let $f : \mathbb{R}^n \to [-\infty, +\infty]$ be an extended real-valued function and let $x \in dom(f)$. Then the following properties hold:

(i) *$J^{2,+}f(x)$ is a convex subset of $\mathbb{R}^n \times S(n)$,*

(ii) *for any $q \in \mathbb{R}^n$, the set $\{Q \in S(n) : (q, Q) \in J^{2,+}f(x)\}$ is a closed convex subset of $S(n)$,*

(iii) *if $f \leq g$ and $f(\widehat{x}) = g(\widehat{x})$ for some $\widehat{x} \in \mathbb{R}^n$, then $J^{2,+}g(\widehat{x}) \subset J^{2,+}f(\widehat{x})$.*

(iv) *if $(q, Q) \in J^{2,+}f(x)$, then $(q, Q') \in J^{2,+}f(x)$ for all $Q' \in S(n)$ such that $Q' \geq Q$. Thus, the set $J^{2,+}f(x)$ is either empty or unbounded.*

Matrix Riccati Equation

Assume $H \in C^2(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and $\partial^+\phi(z) \neq \emptyset$ for all $z \in \mathbb{R}^n$.

Let $\overline{x}$ be an optimal solution of $\mathcal{P}(t_0, x_0)$ and consider a dual arc $\overline{p}$ satisfying $0 \neq \overline{p}(T) \in -\partial^+\phi(\overline{x}(T))$.

From now on set $H_{px}[t] := \nabla^2_{px} H(\overline{x}(t), \overline{p}(t))$, and let $H_{xp}[t], H_{pp}[t], H_{xx}[t]$ be defined analogously.

Riccati Equation : $R(T) = -\nabla^2\phi(\bar{x}(T))$

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

If $V(t, \cdot)$ is C^2 in a neighborhood of $\bar{x}(t)$ for all $t \in [t_0, T]$, then

$$(\nabla_x V(t, \bar{x}(t)), \nabla^2_{xx} V(t, \bar{x}(t)) = (-\bar{p}(t), -R(t))$$

This is a second order sensitivity relation.

Matrix Riccati Equation

Assume $H \in C^2(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and $\partial^+\phi(z) \neq \emptyset$ for all $z \in \mathbb{R}^n$.

Let $\overline{x}$ be an optimal solution of $\mathcal{P}(t_0, x_0)$ and consider a dual arc $\overline{p}$ satisfying $0 \neq \overline{p}(T) \in -\partial^+\phi(\overline{x}(T))$.

From now on set $H_{px}[t] := \nabla^2_{px} H(\overline{x}(t), \overline{p}(t))$, and let $H_{xp}[t], H_{pp}[t], H_{xx}[t]$ be defined analogously.

Riccati Equation : $R(T) = -\nabla^2\phi(\bar{x}(T))$

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

If $V(t,\cdot)$ is C^2 in a neighborhood of $\bar{x}(t)$ for all $t \in [t_0, T]$, then

$$(\nabla_x V(t, \bar{x}(t)), \nabla^2_{xx} V(t, \bar{x}(t)) = (-\bar{p}(t), -R(t))$$

This is a second order sensitivity relation.

Matrix Riccati Equation

Assume $H \in C^2(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and $\partial^+\phi(z) \neq \emptyset$ for all $z \in \mathbb{R}^n$.

Let $\overline{x}$ be an optimal solution of $\mathcal{P}(t_0, x_0)$ and consider a dual arc $\overline{p}$ satisfying $0 \neq \overline{p}(T) \in -\partial^+\phi(\overline{x}(T))$.

From now on set $H_{px}[t] := \nabla^2_{px} H(\overline{x}(t), \overline{p}(t))$, and let $H_{xp}[t], H_{pp}[t], H_{xx}[t]$ be defined analogously.

Riccati Equation : $R(T) = -\nabla^2\phi(\bar{x}(T))$

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

If $V(t, \cdot)$ is C^2 in a neighborhood of $\bar{x}(t)$ for all $t \in [t_0, T]$, then

$$(\nabla_x V(t, \bar{x}(t)), \nabla^2_{xx} V(t, \bar{x}(t)) = (-\bar{p}(t), -R(t))$$

This is a second order sensitivity relation.

Sensitivity Relations Involving Superjets

Theorem

Let $(q, Q) \in J^{2,+}\phi(\overline{x}(T))$, $q \neq 0$ and $\overline{p}(\cdot)$ be the dual arc such that $\overline{p}(T) = -q$. Consider the solution $R(\cdot)$ of

$$\begin{cases} \dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0 \\ R(T) = -Q, \end{cases}$$

defined on $[a, T]$ for some $a \in [t_0, T)$. Then

$$(-\overline{p}(t), -R(t)) \in J^{2,+}_x V(t, \overline{x}(t)) \quad \text{for all } t \in [a, T].$$

Proof is an adaptation of the one in Caroff and HF, TAMS 1996

Sensitivity Relations Involving Superjets

Theorem

Let $(q, Q) \in J^{2,+}\phi(\overline{x}(T))$, $q \neq 0$ *and* $\overline{p}(\cdot)$ *be the dual arc such that* $\overline{p}(T) = -q$. *Consider the solution* $R(\cdot)$ *of*

$$\begin{cases} \dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0 \\ R(T) = -Q, \end{cases}$$

defined on $[a, T]$ *for some* $a \in [t_0, T)$. *Then*

$$(-\overline{p}(t), -R(t)) \in J_x^{2,+}V(t, \overline{x}(t)) \quad \text{for all } t \in [a, T].$$

Proof is an adaptation of the one in Caroff and HF, TAMS 1996

Sensitivity Relations Involving Subjets

Theorem

Let $H \in C^{2,1}_{loc}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ *and for some* $R_0 \in S(n)$

$$(-\overline{p}(t_0), -R_0) \in J^{2,-}_x V(t_0, x_0).$$

If the solution $R(\cdot)$ *of the Riccati equation*

$$\begin{cases} \dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0 \\ R(t_0) = R_0 \end{cases}$$

is well defined on $[t_0, a]$ *for some* $a \in (t_0, T]$*, then*

$$(-\overline{p}(t), -R(t)) \in J^{2,-}_x V(t, \overline{x}(t)) \qquad \textit{for all } t \in [t_0, a].$$

Forward Propagation of Twice Differentiability

Theorem

If $V(t_0,\cdot)$ is twice differentiable at x_0 and the solution $R(\cdot)$ of

$$\begin{cases} \dot R(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0 \\ R(t_0) = -\nabla_{xx}V(t_0,x_0) \end{cases}$$

is well defined on $[t_0,a]$, then $V(t,\cdot)$ is twice differentiable at $\overline{x}(t)$ for any $t \in [t_0,a]$ and $R(t) = -\nabla_{xx}V(t,\overline{x}(t))$.

If ϕ is locally semiconcave, then the interval $[t_0,a]$ can be taken equal to $[t_0,T]$.

A similar result holds true also backward in time.

Reachable Hessian

$Q \in M(n)$ is a reachable Hessian of $V(t,\cdot)$ at x if for some $x_j \to x$, $V(t,\cdot)$ is twice Fréchet differentiable at x_j, and

$$Q = \lim_{j\to\infty} \nabla^2_{xx} V(t, x_j).$$

The set of all reachable Hessians of $V(t,\cdot)$ at x is denoted by $\partial^*_{xx} V(t,x)$. By construction, $\partial^*_{xx} V(t,x)$ is a subset of $S(n)$.

Propagation of Reachable Hessians

Proposition

Let $\partial^+\phi(\overline{x}(T)) \neq \{0\}$ and consider a dual arc $\overline{p}$ satisfying $-\overline{p}(T) \in \partial^+\phi(\overline{x}(T)) \setminus \{0\}$. If $V(t_0, \cdot)$ is of class C^1 in a neighborhood of x_0 and for some $a \in (t_0, T]$ and $Q \in \partial^*_{xx} V(t_0, x_0)$, the Riccati equation

$$\begin{cases} \dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0, \\ R(t_0) = -Q, \end{cases}$$

has a solution $R(\cdot)$ defined on $[t_0, a]$, then, for all $t \in [t_0, a]$,

$$R(t) \in -\partial^*_{xx} V(t, \overline{x}(t)). \quad (1)$$

Moreover, if ϕ is a locally semiconcave function, then $R(\cdot)$ is well defined on $[t_0, T]$, and so (1) is true on the whole interval $[t_0, T]$.

Avoiding Conjugate Times

Assume $\phi \in C^2(\mathbb{R}^n)$ and consider the Riccati equation

$$\begin{cases} \dot{R} + H_{px}[t]R + RH_{xp}[t] + RH_{pp}[t]R + H_{xx}[t] = 0, \\ R(T) = -\nabla^2\phi(\bar{x}(T)). \end{cases}$$

If for some $t_c \in [t_0, T]$, $R(\cdot)$ is well defined on $(t_c, T]$ and $\lim_{t \searrow t_c} \| R(t) \| = +\infty$, then t_c is the conjugate time for $\bar{x}(T)$.

Theorem

Let $\overline{x}$ be optimal for $\mathcal{P}(t_0, x_0)$ with $\nabla\phi(\overline{x}(T)) \neq 0$. If $\partial_x^{-,pr} V(t_0, x_0) \neq \emptyset$, then $R(\cdot)$ is well defined on $[t_0, T]$ and $V(t, \cdot)$ is of class C^2 in a neighborhood of $\overline{x}(t)$ for all $t \in [t_0, T]$.

$\partial_x^{-,pr} V(t, x) \neq \emptyset$ on a dense subset of $x \in \mathbb{R}^n$.

Avoiding Conjugate Times

Assume $\phi \in C^2(\mathbb{R}^n)$ and consider the Riccati equation

$$\begin{cases} \dot{R} + H_{px}[t]R + RH_{xp}[t] + RH_{pp}[t]R + H_{xx}[t] = 0, \\ R(T) = -\nabla^2\phi(\bar{x}(T)). \end{cases}$$

If for some $t_c \in [t_0, T]$, $R(\cdot)$ is well defined on $(t_c, T]$ and $\lim_{t \searrow t_c} \| R(t) \| = +\infty$, then t_c is the conjugate time for $\bar{x}(T)$.

Theorem

Let $\overline{x}$ be optimal for $\mathcal{P}(t_0, x_0)$ with $\nabla\phi(\overline{x}(T)) \neq 0$. If $\partial_x^{-,pr} V(t_0, x_0) \neq \emptyset$, then $R(\cdot)$ is well defined on $[t_0, T]$ and $V(t, \cdot)$ is of class C^2 in a neighborhood of $\overline{x}(t)$ for all $t \in [t_0, T]$.

$\partial_x^{-,pr} V(t, x) \neq \emptyset$ on a dense subset of $x \in \mathbb{R}^n$.

Mayer's Problem under State Constraints

Consider the minimization problem

$$V(t_0, x_0) := \min \left\{ \phi(x(T)) \ : \ x(\cdot) \in S_{[t_0,T]}(x_0), \ x([t_0, T]) \subset K \right\}$$

where $K \subset \mathbb{R}^n$ is nonempty and closed.

Inward Pointing Condition (IPC):

$$\mathrm{co}\, f(y, U) \cap \mathrm{int}\, C_K(y) \neq \emptyset \quad \forall\, y \in \partial K$$

where $C_K(y)$ denotes the Clarke tangent cone to K at y.

If ϕ is locally Lipschitz, then V is locally Lipschitz on $(-\infty, T] \times K$.

Mayer's Problem under State Constraints

Consider the minimization problem

$$V(t_0, x_0) := \min\left\{\phi(x(T)) \ : \ x(\cdot) \in S_{[t_0,T]}(x_0),\ x([t_0, T]) \subset K\right\}$$

where $K \subset \mathbb{R}^n$ is nonempty and closed.

Inward Pointing Condition (IPC):

$$\operatorname{co} f(y, U) \cap \operatorname{int} C_K(y) \neq \emptyset \quad \forall\, y \in \partial K$$

where $C_K(y)$ denotes the Clarke tangent cone to K at y.

If ϕ is locally Lipschitz, then V is locally Lipschitz on $(-\infty, T] \times K$.

Generalized Gradients on Closed Sets

Define partial generalized gradient of V

$$\partial_x V(t,y) \ := \ \mathrm{co}\, \mathrm{Limsup}_{y' \to_{\mathrm{IntK}}\, y} \{\nabla_x V(t,y')\}$$

and generalized gradient of V

$$\partial V(t,y) \ := \ \mathrm{co}\, \mathrm{Limsup}_{(t',y') \to_{[0,T] \times \mathrm{IntK}}\, (t,y)} \{\nabla V(t',y')\, \}$$

In the next result $N_K(y)$ denotes Clarke's normal cone to K at y and $\widehat{V} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\widehat{V}(y) = \begin{cases} -V(t_0, y) & \text{if } y \in K \\ +\infty & \text{otherwise.} \end{cases}$$

Necessary Optimality Conditions

Theorem (Bettiol, Frankowska, Vinter, AMO 2014)

*Let $\bar{x}(.)$ be optimal for the initial condition (t_0, x_0).
Then there exist an arc $p(\cdot)$, a finite positive Borel measure $\mu(\cdot)$ on $[t_0, T]$ and a Borel measurable selection $\nu(t) \in N_K(\bar{x}(t)) \cap B$ μ-a.e. in $[t_0, T]$ such that for $q(s) := p(s) + \int_{[t_0,s]} \nu(\tau) d\mu(\tau)$*

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial H(\bar{x}(t), q(t)) \quad \text{a.e. in } [t_0, T]$$

$-q(T) \in \partial\phi(\bar{x}(T))$, $p(t_0) \in \partial\widehat{V}(\bar{x}(t_0))$ *and the following sensitivity relations hold true for a.e. $t \in [t_0, T]$:*

$$-q(t) \in \partial_x V(t, \bar{x}(t))$$

$$(H(\bar{x}(t), q(t)), -q(t)) \in \partial V(t, \bar{x}(t))$$

Thank you for your attention!