Why don't we move slower?

On the cost of time in the neural control of movement

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IHP, November 26, 2014

Warning

- This is not a non-holonomic talk... not even a mathematical one.
- Application of optimal control to the modelling of human motor control.

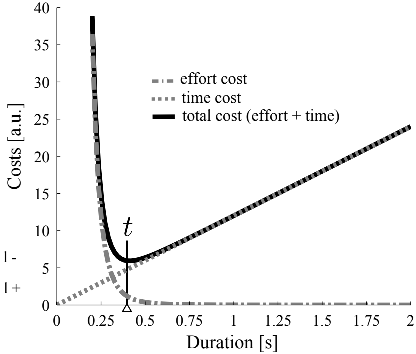
Outline

1. Cost of time in human motions
2. Recovering g
3. Inverse optimal control
4. Experimental results
5. From self paced to slow/fast motions

What is the duration of a natural movement?

(Natural = realize one task, without constraint of time, precision,...)

Usual theory: duration results from the minimization of a compromise between the cost of the time and the cost of the motion.



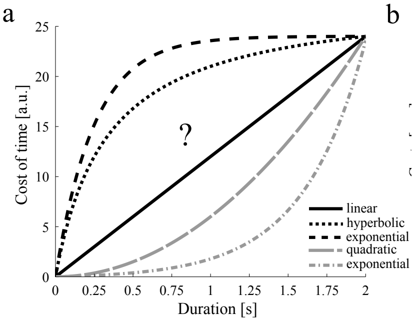
- Cost of the motion (in fixed time):

  solution of an inverse optimal control problem,

  large literature in the case of the arm [Flash, Shadmehr, Berret, ...]

- Cost of the time: lot of different modelling,
  - psychologists → hyperbolic costs (concave functions)
  - economists, behaviourists → exponential costs (convex functions)

  Only interpretations/explanations, no quantitative results.



Modelling

Dynamics of the motion: $\dot{x} = f(x, u)$

Paradigm

Any registered trajectory $x(\cdot)$ from x^0 to x^f is an optimal solution of

$$\min_u \int_0^{t_u} \left(g(t) + L(x_u(t), u(t))\right) dt,$$

among all $u(\cdot)$ defined on $[0, t_u]$ s.t. $x_u(0) = x^0$, $x_u(t_u) = x^f$.

- $\int_0^T g(t)dt$: cost of the time T
- $\int_0^T L(x_u, u)$: cost of the motion in **fixed time** T

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- $\int_0^T g(t)dt$: cost of the time T $\leftarrow$ what we are looking for!
- $\int_0^T L(x_u, u)$: cost of the motion in **fixed time** T

Outline

Necessary condition

Fix x^f. Value function of the problem in **fixed time**:

$$V(t,x) = \inf\{ \int_0^t L(x_u, u) \ : \ \text{for } x_u \text{ joining } x \text{ to } x^f \text{ in time } t \}$$

Set $T =$ movement time from x^0 to x^f. Then

$$T \in \underset{t\in[0,+\infty)}{\operatorname{argmin}} \Big(\int_0^t g(s)ds + V(t,x^0) \Big),$$

and so $g(T) = -\frac{\partial V}{\partial t}(T, x^0)$.

More precisely, $g(T) = -H_0(x^*(T), p^*(T), u^*(T))$ where:

- $H_0(x,p,u) = \langle p, f(x,u)\rangle + L(x,u)$ normal Hamiltonian in fixed time,
- $(x^*(\cdot), u^*(\cdot))$ optimal solution in time T with adjoint vector $p^*(\cdot)$.

<u>Remark:</u> requires two technical assumptions on the fixed time problem:

- existence of minimizers
- no abnormal minimizers (property of the dynamics)

Recovering g from experimental data:

- Fix x^f and choose initial conditions $x^0(a)$, $a \in [a_1, a_2]$.
- Experiments $\longrightarrow$ $T(a) =$ time of motion from $x^0(a)$ to x^f.
- Assume $a \mapsto T(a)$ is invertible and set $a^*(t) = T^{-1}(t)$

[Ex: a amplitude $\Rightarrow T \nearrow$]

Then $$g(t) = -\frac{\partial V}{\partial t}(t, x^0(a^*(t))).$$

Conclusion

Given the cost of motion $L(x, u)$,
the cost of time g can be deduced from simple experiments.

Problems: how to determine $L(x, u)$?
Robustness of the construction of g?

Outline

Inverse optimal control

(Direct) Optimal control problem

Given a dynamic $\dot{x} = f(x,u)$, a cost $C(x_u)$ and a pair of points x_0, x_1, find a trajectory x_{u^*} solution of

$$\inf\{C(x_u) \ : \ x_u \text{ traj. s.t. } x_u(0) = x_0, x_u(T) = x_1\}.$$

Inverse optimal control problem

Given $\dot{x} = f(x,u)$ and a set Γ of trajectories, find a cost $C(x_u)$ such that every $\gamma \in \Gamma$ is solution of

$$\inf\{C(x_u) \ : \ x_u \text{ traj. s.t. } x_u(0) = \gamma(0), x_u(T) = \gamma(T)\}.$$

Applications to analysis/modelling of human motor control (physiology)

$\rightarrow$ looking for optimality principles

Inverse problem: Choose a class $\mathcal{C}$ of reasonable costs and let

$$\Phi : C \in \mathcal{C} \mapsto \Gamma \text{ optimal synthesis.}$$

$$\text{Inverse optimal control problem} = \text{find an inverse } \Phi^{-1}.$$

Well-posed problem?

- Φ injective?
- Continuity (and stability) of Φ^{-1}?

$\rightarrow$ Very few general results:

- Calculus of Variations case [Krupkova, Prince,... 1990-2000's],
- Linear-Quadratic case [Kalmann 64, Nori-Frezza 04],
- numerical methods
  [Mombaur-Laumond 2010, Pauwels-Henrion-Lasserre 2014]

Theoretical result

Dynamics = the one of the 1 doF arm motion (single-input linear system)

Let $\mathcal{SC}$ = set of smooth functions $L(x,u)$ such that

- $\frac{\partial^2 L}{\partial u^2} > 0$ (strict convexity)
- $(0,0)$ unique minimum, $L(0,0) = 0$, and $\frac{\partial^2 L}{\partial u^2}(0,0) = 1$ (normalization)

Theorem

There exists a dense subset Ω s.t. Φ injective on Ω.

(Proof based on Thom transversality)

Open questions:

- continuity of $(\Phi|_\Omega)^{-1}$?
- $\Phi(\Omega)$ dense in $\Phi(\mathcal{SC})$?

Practical point of view

- Linear dynamics: $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$.
- Class of admissible costs = class of quadratic costs,

$$\mathcal{C} = \{L(x,u) = u^T Qu + x^T Rx + 2x^T Su, \quad Q \succ 0,\ L \text{ sym},\ \succeq 0\}.$$

$\rightarrow$ optimal controls in time T of the form $u(t) = K_T(t)x(t)$.

Remark: $\{K_T(\cdot),\ T > 0\}$ uniquely determined by a pair (K_-, K_+), optimal solutions in time T of the form:

$$x(t) = e^{(A+BK_+)t} y_+ + e^{(A+BK_-)(t-T)} y_-.$$

- Hyp: Single input case, i.e. $u \in \mathbb{R}$

Theorem (adapted from Nori-Frezza, 2004), single-input case

For any $L \in \mathcal{C}$, there exists a unique $k \in \mathbb{R}^n$ s.t.

$$\Phi(L) = \Phi\big((u - k^T x)^2\big).$$

Moreover, $(K_-, K_+) \mapsto k$ continuous ($k = -K_+^T$).

$\rightarrow$ the inverse optimal control problem is well-posed.

Application to the computation of g

- Find (K_-, K_+) by identification from experimental data;
- set $k = -K_+^T$ and $L(x,u) = (u - k^T x)^2$;
- compute the function $\frac{\partial V}{\partial t}(t,x)$ using the Hamiltonian;
- set $g(t) = -\frac{\partial V}{\partial t}(t, x^0(a^*(t)))$.

$\rightarrow$ Method robust w.r.t. perturbations of the data and w.r.t. the choice of cost.

Quantitative result:

Assume $x^f = 0$ and $x^0(a) = a x^0(1)$. Then:

- $\frac{\partial V}{\partial t}(T, x^0) = -u_a(T)^2$, where $u_a(\cdot)$ = optimal control from $x^0(a)$ to x^f in time T.
- $u_a(T) = a\nu(T)$, where

$$\nu(T) = (K_+ - K_-)\left(e^{-(A+BK_-)T} - e^{-(A+BK_+)T}\right)^{-1} x^0(1).$$

$$\Rightarrow \qquad \boxed{g(t) = \nu(t)^2 a^*(t)^2}$$

Outline

Two kind of motions:

- Pointing motions of the arm in a horizontal plane (1 degree of freedom),
- Saccadic eye movements.

In both cases:

- the dynamic is of the form:

$$\theta^{(n)} + c_{n-1}\theta^{(n-1)} + \cdots + c_0\theta = u,$$

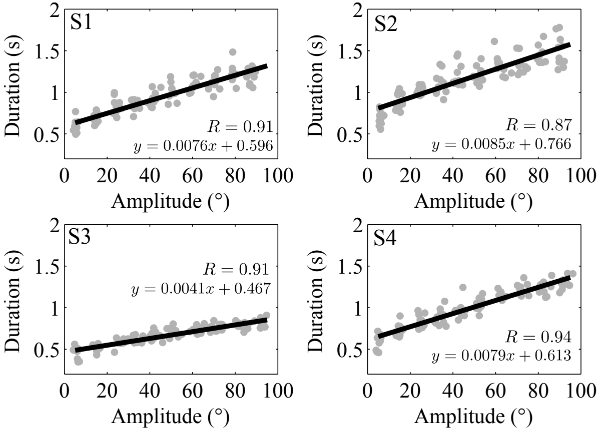
$\rightarrow$ linear with a state $x = (\theta, \dot{\theta}, \ldots, \theta^{(n-1)})$ ($n = 2$ or 3 in general)

- Initial and final states are equilibria, typically:

$$x^0(a) = (a, 0, \ldots, 0) \quad \text{and} \quad x^f = 0.$$

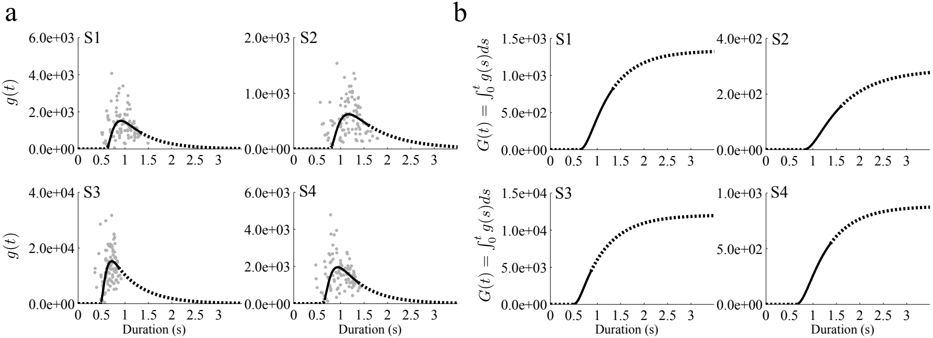
Observations: For arm pointing motions, $T(a)$ affine function

$$\Leftrightarrow \qquad a^*(t) = \alpha t - \beta.$$



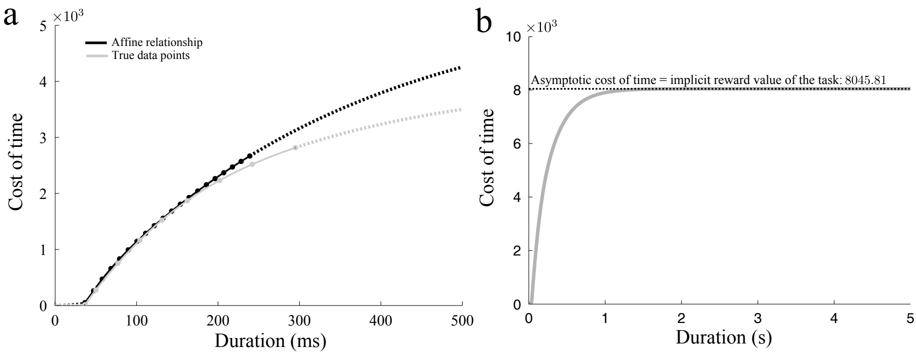
Computation of g

For arm pointing motions:
cost of time neither hyperbolic nor exponential...



Computation of g

For eye saccadic movements:
cost of time hyperbolic, as expected



Outline

Interpretation

Form of the infinitesimal cost (with $x^f = 0$):

$$g(t) + u^2 + x^T R x \quad [+ \text{ eventually mixed terms } x^T S u]$$

- $g(t) =$ cost of the time
- $u^2 =$ cost of the particular motion (objective cost)
- $x^T R x =$ cost related to the target (penalization)

$\Rightarrow$ change of the instructions ("faster/slower") $=$ change of R

Example: for 1 doF motions,

$$x^T R x = s_0 \theta^2 + s_1 \dot{\theta}^2 + \cdots + s_{n-1} (\theta^{(k-1)})^2$$

$\rightarrow$ "Go quickly to the goal" $=$ increase s_0

$\rightarrow$ "Go slowly" $=$ increase s_1

Let $T_{s_0,\dots,s_{n-1}}(a)$ = duration of an optimal solution between $x^0(a)$ and x^f.

Lemma (Sensitivity analysis)

$$\begin{cases} s_0 \mapsto T_s(a) \searrow \\ s_1 \mapsto T_s(a) \nearrow \end{cases}$$

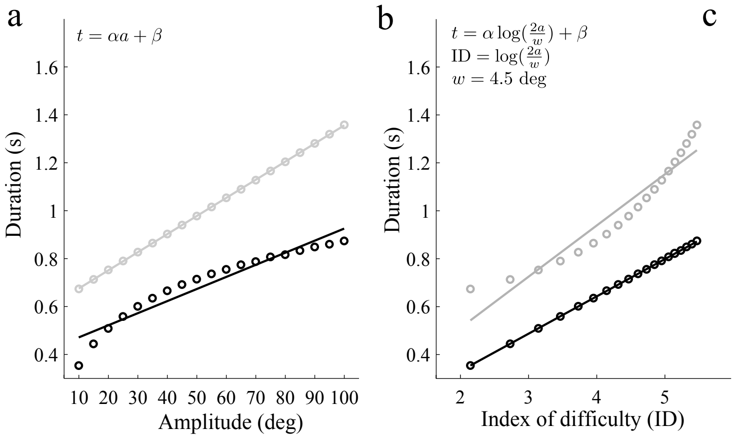
Moreover, when $s_0 \to \infty$, $\theta^*(t) \sim aCe^{-\lambda t}$ for $t \in [T_s(a)/2, T_s(a))$

Consequence: if the target centered at 0 has a width w, the motion stops at a time T s.t. $\theta(T) = w/2$

$$\Rightarrow \qquad T \sim \frac{1}{\lambda}\log(\frac{a}{w}) + \frac{\log 2C}{\lambda}.$$

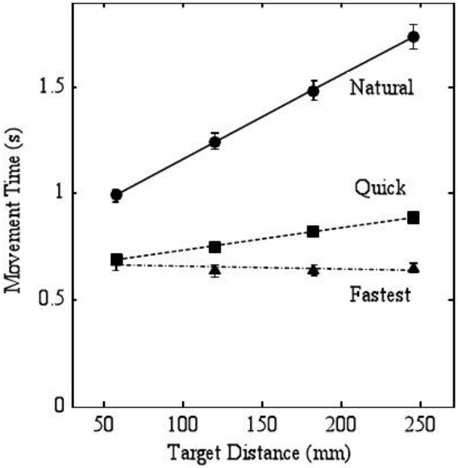
$\rightarrow$ Fitts's law

Self-paced (gray) or maximal speed (black) movements:
duration $=$ affine w.r.t. amplitude a or ID$= a/w$



Aim

Show that the above interpretation may explain the picture below.



Data from [Young 2009]:

time vs amplitude
with constant ID$= a/w$