Strong local optimality of singular trajectories

Gianna Stefani

Dipartimento di Matematica e Informatica, Università di Firenze

Nonholonomic mechanics and optimal control

Institut Henri Poincaré, November 25-28, 2014

The optimal control problem

Minimize $J(\xi, u, T)$ subject to

$$\begin{cases} \dot{\xi}(t) = (f_0 + \sum_{i=1}^m u_i(t) f_i) \circ \xi(t) \\ \xi(0) \in N_0, \quad \xi(T) \in N_f \\ \mathbf{u} = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m, \ \operatorname{int} U \neq \emptyset. \end{cases}$$

state space = M (n-dimensional), N_0, N_f - sub-manifolds

C^∞ data, L^∞ control maps

The focus will be on the case when the controlled vector fields $f_1, \dots, f_m$ generate a non involutive Lie Algebra

joint work with F.C. Chittaro

for what is written I refer to *http://arxiv.org/abs/1404.7336*

partial results are in *CDC 2013*

The optimal control problem

Minimize $J(\xi, u, T)$ subject to

$$\begin{cases} \dot{\xi}(t) = (f_0 + \sum_{i=1}^m u_i(t) f_i) \circ \xi(t) \\ \xi(0) \in N_0, \quad \xi(T) \in N_f \\ \\ \mathbf{u} = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m, \ \operatorname{int} U \neq \emptyset. \end{cases}$$

state space = M (n-dimensional), N_0, N_f - sub-manifolds

C^∞ data, L^∞ control maps

The focus will be on the case when the controlled vector fields $f_1, \ldots, f_m$ generate a non involutive Lie Algebra

joint work with F.C. Chittaro

for what is written I refer to *http://arxiv.org/abs/1404.7336*

partial results are in *CDC 2013*

The optimal control problem

Minimize $J(\xi, u, T)$ subject to

$$\begin{cases} \dot{\xi}(t) = (f_0 + \sum_{i=1}^m u_i(t) f_i) \circ \xi(t) \\ \xi(0) \in N_0, \quad \xi(T) \in N_f \\ \\ \mathbf{u} = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m, \ \operatorname{int} U \neq \emptyset. \end{cases}$$

state space = M (n-dimensional), N_0, N_f - sub-manifolds

C^∞ data, L^∞ control maps

The focus will be on the case when the controlled vector fields $f_1, \ldots, f_m$ generate a non involutive Lie Algebra

joint work with F.C. Chittaro

for what is written I refer to *http://arxiv.org/abs/1404.7336*

partial results are in *CDC 2013*

The cost

- Mayer problem on $[0, \widehat{T}]$ fixed

$$\text{minimize} \quad c_0(\xi(0)) + c_f(\xi(\widehat{T}))$$

- minimum time problem

$$\text{minimize the final time} \quad T$$

Reference couple satisfying PMP

$\widehat{\mathbf{u}} \colon [0, \widehat{T}] \to U$ reference control, $\widehat{\xi} \colon [0, \widehat{T}] \to M$ reference trajectory

an hat above denotes reference objects

Reference vector field

$\widehat{f}_t = f_0 + \sum_{i=1}^{m} \widehat{u}_i(t) f_i$

The cost

- Mayer problem on $[0,\widehat{T}]$ fixed

$$\text{minimize} \quad c_0(\xi(0)) + c_f(\xi(\widehat{T}))$$

- minimum time problem

$$\text{minimize the final time} \quad T$$

Reference couple satisfying PMP

$\widehat{\mathbf{u}}\colon [0,\widehat{T}] \to U$ reference control, $\widehat{\xi}\colon [0,\widehat{T}] \to M$ reference trajectory

an hat above denotes reference objects

Reference vector field

$\widehat{f}_t = f_0 + \sum_{i=1}^{m} \widehat{u}_i(t) f_i$

The cost

- Mayer problem on $[0, \widehat{T}]$ fixed

$$\text{minimize} \quad c_0(\xi(0)) + c_f(\xi(\widehat{T}))$$

- minimum time problem

$$\text{minimize the final time} \quad T$$

Reference couple satisfying PMP

$\widehat{\mathbf{u}} \colon [0, \widehat{T}] \to U$ reference control, $\widehat{\xi} \colon [0, \widehat{T}] \to M$ reference trajectory

an hat above denotes reference objects

Reference vector field

$\widehat{f}_t = f_0 + \sum_{i=1}^{m} \widehat{u}_i(t) f_i$

Aim of the talk

To give necessary conditions and sufficient conditions for the

strong local optimality of $(\widehat{\mathbf{u}}, \widehat{\xi})$ in the case when the couple is

totally singular ($\widehat{\mathbf{u}}(t) \in \operatorname{int} U$)

partially singular (only some control maps take interior values)

Strong local optimality (SLO)

Roughly speaking:

$\widehat{\xi}$ is a strong local minimizer if it is a minimizer w.r.t. admissible trajectories ξ *"near in graph"* to $\widehat{\xi}$ independently of the values of the control functions

SLO is a strong property : the optimality has to be *independent of the control set*, hence the sufficient conditions have to hold true also for unbounded control sets

Aim of the talk

To give necessary conditions and sufficient conditions for the

strong local optimality of $(\widehat{\mathbf{u}}, \widehat{\xi})$ in the case when the couple is

totally singular ($\widehat{\mathbf{u}}(t) \in \operatorname{int} U$)

partially singular (only some control maps take interior values)

Strong local optimality (SLO)

Roughly speaking:

$\widehat{\xi}$ is a strong local minimizer if it is a minimizer w.r.t. admissible trajectories ξ *"near in graph"* to $\widehat{\xi}$ independently of the values of the control functions

SLO is a strong property : the optimality has to be *independent of the control set*, hence the sufficient conditions have to hold true also for unbounded control sets

Aim of the talk

To give necessary conditions and sufficient conditions for the

strong local optimality of $(\widehat{\mathbf{u}}, \widehat{\xi})$ in the case when the couple is

totally singular ($\widehat{\mathbf{u}}(t) \in \operatorname{int} U$)

partially singular (only some control maps take interior values)

Strong local optimality (SLO)

Roughly speaking:

$\widehat{\xi}$ is a strong local minimizer if it is a minimizer w.r.t. admissible trajectories ξ *"near in graph"* to $\widehat{\xi}$ independently of the values of the control functions

SLO is a strong property : the optimality has to be *independent of the control set*, hence the sufficient conditions have to hold true also for unbounded control sets

Partially singular trajectories

the control set

U is a box given by $|u_i| \le a_i$ *(possibly unbounded in some direction)*

the reference control map

the reference control maps are either bang-bang or singular

$$|\widehat{u}_i(t)| < a_i\,,\; i = 1, \dots, m_1 \qquad \text{and} \qquad |\widehat{u}_i(t)| = a_i\,,\; i = m_1 + 1, \dots, m$$

the subsystem

I consider also the subsystem where only the singular controls vary :

$$\dot{\xi}(t) = \left(\widehat{f}_t + \sum_{i=1}^{m_1} u_i(t) f_i\right) \circ \xi(t) \quad \text{with} \quad \mathbf{u_1} \in U_1 \subset \mathbb{R}^{m_1},\; \operatorname{int} U_1 \neq \emptyset.$$

Partially singular trajectories

the control set

U is a box given by $|u_i| \leq a_i$ *(possibly unbounded in some direction)*

the reference control map

the reference control maps are either bang-bang or singular

$$|\widehat{u}_i(t)| < a_i\,,\ i = 1, \ldots, m_1 \qquad \text{and} \qquad |\widehat{u}_i(t)| = a_i\,,\ i = m_1 + 1, \ldots, m$$

the subsystem

I consider also the subsystem where only the singular controls vary :

$$\dot{\xi}(t) = \left(\widehat{f}_t + \sum_{i=1}^{m_1} u_i(t) f_i\right) \circ \xi(t) \quad \text{with} \quad \mathbf{u_1} \in U_1 \subset \mathbb{R}^{m_1},\ \operatorname{int} U_1 \neq \emptyset.$$

Partially singular trajectories

the control set

U is a box given by $|u_i| \leq a_i$ *(possibly unbounded in some direction)*

the reference control map

the reference control maps are either bang-bang or singular

$$|\widehat{u}_i(t)| < a_i\,, \ i = 1, \ldots, m_1 \qquad \text{and} \qquad |\widehat{u}_i(t)| = a_i\,, \ i = m_1 + 1, \ldots, m$$

the subsystem

I consider also the subsystem where only the singular controls vary :

$$\dot{\xi}(t) = \left(\widehat{f}_t + \sum_{i=1}^{m_1} u_i(t) f_i\right) \circ \xi(t) \quad \text{with} \quad \mathbf{u_1} \in U_1 \subset \mathbb{R}^{m_1},\ \operatorname{int} U_1 \neq \emptyset.$$

Optimality conditions with unbounded controls

totally singular: $U = \mathbb{R}^m$

partially singular: consider the subsystem with $U_1 = \mathbb{R}^{m_1}$

From one hand unbounded controls are the natural setting for strong local optimality and justify the gap between necessary and sufficient conditions when the control set is bounded. This gap appears only when the Lie algebra generated by the controlled vector fields is not involutive.

From the other hand unbounded controls are possibly related to ”optimal trajectories” with jumps.

Necessary conditions

High Order Maximum Principle will be given

Sufficient conditions

The sufficient condition are given in terms of regularity conditions and of the coercivity of a suitable quadratic form.

Optimality conditions with unbounded controls

totally singular: $U = \mathbb{R}^m$

partially singular: consider the subsystem with $U_1 = \mathbb{R}^{m_1}$

From one hand unbounded controls are the natural setting for strong local optimality and justify the gap between necessary and sufficient conditions when the control set is bounded. This gap appears only when the Lie algebra generated by the controlled vector fields is not involutive.

From the other hand unbounded controls are possibly related to ”optimal trajectories” with jumps.

Necessary conditions

High Order Maximum Principle will be given

Sufficient conditions

The sufficient condition are given in terms of regularity conditions and of the coercivity of a suitable quadratic form.

Optimality conditions with unbounded controls

totally singular: $U = \mathbb{R}^m$

partially singular: consider the subsystem with $U_1 = \mathbb{R}^{m_1}$

From one hand unbounded controls are the natural setting for strong local optimality and justify the gap between necessary and sufficient conditions when the control set is bounded. This gap appears only when the Lie algebra generated by the controlled vector fields is not involutive.

From the other hand unbounded controls are possibly related to ”optimal trajectories” with jumps.

Necessary conditions

High Order Maximum Principle will be given

Sufficient conditions

The sufficient condition are given in terms of regularity conditions and of the coercivity of a suitable quadratic form.

Optimality conditions with unbounded controls

totally singular: $U = \mathbb{R}^m$

partially singular: consider the subsystem with $U_1 = \mathbb{R}^{m_1}$

From one hand unbounded controls are the natural setting for strong local optimality and justify the gap between necessary and sufficient conditions when the control set is bounded. This gap appears only when the Lie algebra generated by the controlled vector fields is not involutive.

From the other hand unbounded controls are possibly related to ”optimal trajectories” with jumps.

Necessary conditions

High Order Maximum Principle will be given

Sufficient conditions

The sufficient condition are given in terms of regularity conditions and of the coercivity of a suitable quadratic form.

Optimality conditions with unbounded controls

totally singular: $U = \mathbb{R}^m$

partially singular: consider the subsystem with $U_1 = \mathbb{R}^{m_1}$

From one hand unbounded controls are the natural setting for strong local optimality and justify the gap between necessary and sufficient conditions when the control set is bounded. This gap appears only when the Lie algebra generated by the controlled vector fields is not involutive.

From the other hand unbounded controls are possibly related to ”optimal trajectories” with jumps.

Necessary conditions

High Order Maximum Principle will be given

Sufficient conditions

The sufficient condition are given in terms of regularity conditions and of the coercivity of a suitable quadratic form.

Notation

The symplectic structure on $\pi : T^*M \to M$ allows to lift vector fields on M to functions and vector fields on T^*M

Hamiltonians associated to f_i , $\quad i = 0, 1, \ldots, m$

$F_i : \ell \in T^*M \mapsto \langle \ell\,,\ f_i(\pi\ell) \rangle \in \mathbb{R}$ Hamiltonian
$\vec{F}_i : T^*M \to TT^*M$ Hamiltonian vector field

Reference Hamiltonian (possibly time-dependent)

$\widehat{F}_t = F_0 + \sum_{i=1}^m \widehat{u}_i(t) F_i$ and $\vec{\widehat{F}}_t = \vec{F}_0 + \sum_{i=1}^m \widehat{u}_i(t) \vec{F}_i$

Maximized Hamiltonian

$F^{\max} : \ell \mapsto \sup_{\mathbf{u} \in U} \left(F_0 + \sum_{i=1}^m u_i F_i \right)(\ell)$

Notation

The symplectic structure on $\pi\colon T^*M \to M$ allows to lift vector fields on M to functions and vector fields on T^*M

Hamiltonians associated to $f_i\,, \quad i = 0, 1, \ldots, m$

$F_i : \ell \in T^*M \mapsto \langle \ell\,,\ f_i(\pi\ell) \rangle \in \mathbb{R}$ Hamiltonian
$\vec{F}_i : T^*M \to TT^*M$ Hamiltonian vector field

Reference Hamiltonian (possibly time-dependent)

$\widehat{F}_t = F_0 + \sum_{i=1}^m \widehat{u}_i(t) F_i$ and $\vec{\widehat{F}}_t = \vec{F}_0 + \sum_{i=1}^m \widehat{u}_i(t) \vec{F}_i$

Maximized Hamiltonian

$F^{\max} : \ell \mapsto \sup_{\mathbf{u} \in U} \left(F_0 + \sum_{i=1}^m u_i F_i \right)(\ell)$

Notation

The symplectic structure on $\pi\colon T^*M \to M$ allows to lift vector fields on M to functions and vector fields on T^*M

Hamiltonians associated to $f_i\,,\quad i = 0, 1, \ldots, m$

$F_i : \ell \in T^*M \mapsto \langle \ell\,,\ f_i(\pi\ell) \rangle \in \mathbb{R}$ Hamiltonian
$\vec{F_i} : T^*M \to TT^*M$ Hamiltonian vector field

Reference Hamiltonian (possibly time-dependent)

$\widehat{F}_t = F_0 + \sum_{i=1}^m \widehat{u}_i(t) F_i$ and $\vec{\widehat{F}}_t = \vec{F_0} + \sum_{i=1}^m \widehat{u}_i(t) \vec{F_i}$

Maximized Hamiltonian

$F^{\max} : \ell \mapsto \sup_{\mathbf{u} \in U} \left(F_0 + \sum_{i=1}^m u_i F_i \right)(\ell)$

Pontryagin Maximum Principle

if $\widehat{\xi}$ is a minimizer then there are $p_0 \in \{0, 1\}$ and an absolutely continuous solution $\widehat{\lambda}\colon [0, \widehat{T}] \to T^*M$ of

$$\dot{\lambda}(t) = \vec{\widehat{F}}_t(\lambda(t))$$

such that $\pi\widehat{\lambda}(t) = \widehat{\xi}(t)$, p_0 and $\widehat{\lambda}$ are not both zero, and

transversality

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = p_0 dc_0(\widehat{\xi}(0)), \quad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = -p_0 dc_f(\widehat{\xi}(\widehat{T}))$ Mayer

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = 0 \qquad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = 0$ minimum time

maximality

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t)$ Mayer

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t) \equiv p_0$ minimum time

$\widehat{\lambda}$ is a Pontryagin extremal and it is also called the adjont convector

Pontryagin Maximum Principle

if $\widehat{\xi}$ is a minimizer then there are $p_0 \in \{0,1\}$ and an absolutely continuous solution $\widehat{\lambda}$: $[0,\widehat{T}] \to T^*M$ of

$$\dot{\lambda}(t) = \vec{\widehat{F}}_t(\lambda(t))$$

such that $\pi\widehat{\lambda}(t) = \widehat{\xi}(t)$, p_0 and $\widehat{\lambda}$ are not both zero, and

transversality

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = p_0 dc_0(\widehat{\xi}(0)), \quad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = -p_0 dc_f(\widehat{\xi}(\widehat{T}))$ Mayer

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = 0 \qquad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = 0$ minimum time

maximality

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t)$ Mayer

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t) \equiv p_0$ minimum time

$\widehat{\lambda}$ is a Pontryagin extremal and it is also called the adjont convector

Pontryagin Maximum Principle

if $\widehat{\xi}$ is a minimizer then there are $p_0 \in \{0, 1\}$ and an absolutely continuous solution $\widehat{\lambda}\colon [0, \widehat{T}] \to T^*M$ of

$$\dot{\lambda}(t) = \vec{\widehat{F}}_t(\lambda(t))$$

such that $\pi\widehat{\lambda}(t) = \widehat{\xi}(t)$, p_0 and $\widehat{\lambda}$ are not both zero, and

transversality

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = p_0 dc_0(\widehat{\xi}(0)), \quad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = -p_0 dc_f(\widehat{\xi}(\widehat{T}))$ Mayer

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = 0 \qquad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = 0$ minimum time

maximality

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t)$ Mayer

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t) \equiv p_0$ minimum time

$\widehat{\lambda}$ is a Pontryagin extremal and it is also called the adjont convector

Pontryagin Maximum Principle

if $\widehat{\xi}$ is a minimizer then there are $p_0 \in \{0,1\}$ and an absolutely continuous solution $\widehat{\lambda}\colon [0,\widehat{T}] \to T^*M$ of

$$\dot{\lambda}(t) = \vec{\widehat{F}}_t(\lambda(t))$$

such that $\pi\widehat{\lambda}(t) = \widehat{\xi}(t)$, p_0 and $\widehat{\lambda}$ are not both zero, and

transversality

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = p_0 dc_0(\widehat{\xi}(0)), \quad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = -p_0 dc_f(\widehat{\xi}(\widehat{T}))$ Mayer

$\widehat{\lambda}(0)|_{T_{\widehat{\xi}(0)}N_0} = 0 \qquad \widehat{\lambda}(\widehat{T})|_{T_{\widehat{\xi}(\widehat{T})}N_f} = 0$ minimum time

maximality

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t)$ Mayer

$\widehat{F}_t \circ \widehat{\lambda}(t) = F^{\max} \circ \widehat{\lambda}(t) \equiv p_0$ minimum time

$\widehat{\lambda}$ is a Pontryagin extremal and it is also called the adjont convector

Consequences for singular trajectories

totally singular trajectories

$$F_i \circ \widehat{\lambda}(t) \equiv 0 \,, \qquad i = 1, \dots, m$$

$$F_{0i} \circ \widehat{\lambda}(t) := \{F_0, F_i\} \circ \widehat{\lambda}(t) \equiv 0 \,, \qquad i = 1, \dots, m$$

partially singular trajectories

$$F_i \circ \widehat{\lambda}(t) \equiv 0 \,, \qquad i = 1, \dots, m_1$$

$$F_{0i} \circ \widehat{\lambda}(t) := \{F_0, F_i\} \circ \widehat{\lambda}(t) \equiv 0 \,, \qquad i = 1, \dots, m_1$$

Consequences for singular trajectories

totally singular trajectories

$$F_i \circ \widehat{\lambda}(t) \equiv 0\,, \qquad i = 1, \ldots, m$$

$$F_{0i} \circ \widehat{\lambda}(t) := \{F_0, F_i\} \circ \widehat{\lambda}(t) \equiv 0\,, \qquad i = 1, \ldots, m$$

partially singular trajectories

$$F_i \circ \widehat{\lambda}(t) \equiv 0\,, \qquad i = 1, \ldots, m_1$$

$$F_{0i} \circ \widehat{\lambda}(t) := \{F_0, F_i\} \circ \widehat{\lambda}(t) \equiv 0\,, \qquad i = 1, \ldots, m_1$$

Further necessary conditions

Goh Condition

$$\{F_i, F_j\} \circ \widehat{\lambda}(t) = \langle \widehat{\lambda}(t), [f_i, f_j](\widehat{\xi}(t)) \rangle = 0, \quad i, j = 1, \ldots, m$$

set $m = m_1$ for the partially singular case

Generalized Legiandre Condition

the quadratic form on $\mathbb{R}^m$

$$\mathrm{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, \widehat{F}_t\}\} \circ \widehat{\lambda}(t) \leq 0 \qquad \text{a.e. } t \in [0, \widehat{T}]$$

set $m = m_1$ for the partially singular case

Further necessary conditions

Goh Condition

$$\{F_i, F_j\} \circ \widehat{\lambda}(t) = \langle \widehat{\lambda}(t), [f_i, f_j](\widehat{\xi}(t)) \rangle = 0, \quad i, j = 1, \dots, m$$

set $m = m_1$ for the partially singular case

Generalized Legiandre Condition

the quadratic form on $\mathbb{R}^m$

$$\mathbf{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, \widehat{F}_t\}\} \circ \widehat{\lambda}(t) \leq 0 \qquad \text{a.e. } t \in [0, \widehat{T}]$$

set $m = m_1$ for the partially singular case

High Order Maximum Principle

Suppose that $\widehat{\xi}$ is a totally (partially) singular minimizer for the problem with $U = \mathbb{R}^m \quad (U_1 = \mathbb{R}^{m_1})$, then

there is an adjoint convector $\widehat{\lambda}$ which satisfies the previous properties and

HOGC: $\quad F \circ \widehat{\lambda}(t) \equiv 0 \quad \forall f \in \text{Lie}\{f_1, \ldots, f_m\} \quad (\text{Lie}\{f_1, \ldots, f_{m_1}\})$

The result was known for $\widehat{\mathbf{u}} \in C^\infty$ and we proved for $\widehat{\mathbf{u}} \in L^\infty$ in the minimum time problem.

The proof is based on high order cones of needle-like variations with "good properties" and we used a result by R.M.Bianchini which apply both to Mayer and minimum-time problems.

Open problem

Prove other high order conditions, possibly for $\widehat{\mathbf{u}} \in L^1$

High Order Maximum Principle

Suppose that $\widehat{\xi}$ is a totally (partially) singular minimizer for the problem with $U = \mathbb{R}^m$ $\quad (U_1 = \mathbb{R}^{m_1}$), then

there is an adjoint convector $\widehat{\lambda}$ which satisfies the previous properties and

HOGC: $\quad F \circ \widehat{\lambda}(t) \equiv 0 \quad \forall\, f \in \text{Lie}\{f_1, \ldots, f_m\} \quad (\text{Lie}\{f_1, \ldots, f_{m_1}\})$

The result was known for $\widehat{\mathbf{u}} \in C^\infty$ and we proved for $\widehat{\mathbf{u}} \in L^\infty$ in the minimum time problem.

The proof is based on high order cones of needle-like variations with ”good properties” and we used a result by R.M.Bianchini which apply both to Mayer and minimum-time problems.

Open problem

Prove other high order conditions, possibly for $\widehat{\mathbf{u}} \in L^1$

High Order Maximum Principle

Suppose that $\widehat{\xi}$ is a totally (partially) singular minimizer for the problem with $U = \mathbb{R}^m$ $(U_1 = \mathbb{R}^{m_1})$, then

there is an adjoint convector $\widehat{\lambda}$ which satisfies the previous properties and

HOGC: $F \circ \widehat{\lambda}(t) \equiv 0 \quad \forall\, f \in \text{Lie}\{f_1, \ldots, f_m\} \quad (\text{Lie}\{f_1, \ldots, f_{m_1}\})$

The result was known for $\widehat{\mathbf{u}} \in C^\infty$ and we proved for $\widehat{\mathbf{u}} \in L^\infty$ in the minimum time problem.

The proof is based on high order cones of needle-like variations with ”good properties” and we used a result by R.M.Bianchini which apply both to Mayer and minimum-time problems.

Open problem

Prove other high order conditions, possibly for $\widehat{\mathbf{u}} \in L^1$

High Order Maximum Principle

Suppose that $\widehat{\xi}$ is a totally (partially) singular minimizer for the problem with $U = \mathbb{R}^m$ $\quad (U_1 = \mathbb{R}^{m_1}$), then

there is an adjoint convector $\widehat{\lambda}$ which satisfies the previous properties and

HOGC: $\quad F \circ \widehat{\lambda}(t) \equiv 0 \quad \forall\, f \in \text{Lie}\{f_1, \dots, f_m\} \quad (\text{Lie}\{f_1, \dots, f_{m_1}\})$

The result was known for $\widehat{\mathbf{u}} \in C^\infty$ and we proved for $\widehat{\mathbf{u}} \in L^\infty$ in the minimum time problem.

The proof is based on high order cones of needle-like variations with "good properties" and we used a result by R.M.Bianchini which apply both to Mayer and minimum-time problems.

Open problem

Prove other high order conditions, possibly for $\widehat{\mathbf{u}} \in L^1$

From High Order Maximum Principle

by deriving HOGC along the adjoint covector

$\{F_0, F\} \circ \widehat{\lambda}(t) \equiv 0, \quad \forall\, f \in \text{Lie}\{f_1, \ldots, f_m\} := \mathcal{L}$

Generalized Legiandre Condition

for $\ell \in T^*M$ define the Generalized Legiandre quadratic form on $\mathbb{R}^m$

$$\mathbb{L}(\ell) : \mathbf{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, F_0\}\}(\ell)$$

the GLC becomes

$$\mathbb{L} \circ \widehat{\lambda}(t) \leq 0 \qquad t \in [0, \widehat{T}]$$

partially singular case

set $m = m_1$ in the above equations

From High Order Maximum Principle

by deriving HOGC along the adjoint covector

$$\{F_0, F\} \circ \widehat{\lambda}(t) \equiv 0, \quad \forall\, f \in \text{Lie}\{f_1, \ldots, f_m\} := \mathcal{L}$$

Generalized Legiandre Condition

for $\ell \in T^*M$ define the Generalized Legiandre quadratic form on $\mathbb{R}^m$

$$\mathbb{L}(\ell) : \mathbf{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, F_0\}\}(\ell)$$

the GLC becomes

$$\mathbb{L} \circ \widehat{\lambda}(t) \leq 0 \qquad t \in [0, \widehat{T}]$$

partially singular case

set $m = m_1$ in the above equations

From High Order Maximum Principle

by deriving HOGC along the adjoint covector

$\{F_0, F\} \circ \widehat{\lambda}(t) \equiv 0, \quad \forall\, f \in \mathrm{Lie}\{f_1, \ldots, f_m\} := \mathcal{L}$

Generalized Legiandre Condition

for $\ell \in T^*M$ define the Generalized Legiandre quadratic form on $\mathbb{R}^m$

$$\mathbb{L}(\ell) : \mathbf{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, F_0\}\}(\ell)$$

the GLC becomes

$$\mathbb{L} \circ \widehat{\lambda}(t) \leq 0 \qquad t \in [0, \widehat{T}]$$

partially singular case

set $m = m_1$ in the above equations

Sufficient conditions

The written results concern the minimum-time problem and a totally singular arc.

The Mayer problem is now in progress.

Partially singular trajectories

The conditions apply when the bang reference controls do not switch:

$\widehat{f}_t = f_0 + \sum_{i=m_1+1}^{m} (\pm a_i) f_i + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i := \tilde{f}_0 + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i.$

The general case is under study

The sufficient conditions for $\widehat{\xi}$ to be a strong local minimizer of the subsystem

$\dot{\xi}(t) = \left(\tilde{f}_0 + \sum_{i=1}^{m_1} u_i(t) f_i \right) \circ \xi(t)$

give the strong-local optimality of $\widehat{\xi}$ w.r.t. the admissible trajectories of the original system.

The result can be obtained thanks to the Hamiltonian approach

Sufficient conditions

The written results concern the minimum-time problem and a totally singular arc.

The Mayer problem is now in progress.

Partially singular trajectories

The conditions apply when the bang reference controls do not switch:

$$\widehat{f}_t = f_0 + \sum_{i=m_1+1}^{m} (\pm a_i) f_i + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i := \tilde{f}_0 + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i.$$

The general case is under study

The sufficient conditions for $\widehat{\xi}$ to be a strong local minimizer of the subsystem

$$\dot{\xi}(t) = \left(\tilde{f}_0 + \sum_{i=1}^{m_1} u_i(t) f_i \right) \circ \xi(t)$$

give the strong-local optimality of $\widehat{\xi}$ w.r.t. the admissible trajectories of the original system.

The result can be obtained thanks to the Hamiltonian approach

Sufficient conditions

The written results concern the minimum-time problem and a totally singular arc.

The Mayer problem is now in progress.

Partially singular trajectories

The conditions apply when the bang reference controls do not switch:

$$\widehat{f}_t = f_0 + \sum_{i=m_1+1}^{m} (\pm a_i) f_i + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i := \tilde{f}_0 + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i.$$

The general case is under study

The sufficient conditions for $\widehat{\xi}$ to be a strong local minimizer of the subsystem

$$\dot{\xi}(t) = \left(\tilde{f}_0 + \sum_{i=1}^{m_1} u_i(t) f_i \right) \circ \xi(t)$$

give the strong-local optimality of $\widehat{\xi}$ w.r.t. the admissible trajectories of the original system.

The result can be obtained thanks to the Hamiltonian approach

Sufficient conditions

The written results concern the minimum-time problem and a totally singular arc.

The Mayer problem is now in progress.

Partially singular trajectories

The conditions apply when the bang reference controls do not switch:

$$\widehat{f_t} = f_0 + \sum_{i=m_1+1}^{m} (\pm a_i) f_i + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i := \tilde{f}_0 + \sum_{i=1}^{m_1} \widehat{u}_i(t) f_i.$$

The general case is under study

The sufficient conditions for $\widehat{\xi}$ to be a strong local minimizer of the subsystem

$$\dot{\xi}(t) = \left(\tilde{f}_0 + \sum_{i=1}^{m_1} u_i(t) f_i \right) \circ \xi(t)$$

give the strong-local optimality of $\widehat{\xi}$ w.r.t. the admissible trajectories of the original system.

The result can be obtained thanks to the Hamiltonian approach

From now on consider the system

$$\begin{cases} \dot{\xi}(t) = (f_0 + \sum_{i=1}^{m} u_i(t) f_i) \circ \xi(t) \\ \xi(0) \in N_0, \quad \xi(T) \in N_f, \quad \mathbf{u} \in \mathbb{R}^m \end{cases}$$

with the following *regularity assumption*

RA1 $\mathcal{L}$ has constant dimension $r \geq m$

$I_x :=$ integral manifold of $\mathcal{L}$ through x

For the minimum time problem require also

$$N_0 \subset I_{\widehat{\xi}(0)} \qquad N_T \subset I_{\widehat{\xi}(\widehat{T})}$$

For the partially singular case set

$$f_0 = \tilde{f}_0 \qquad \text{and} \qquad m = m_1$$

Regularity assumptions I

The adjoint convector $\widehat{\lambda} : [0, \widehat{T}] \to T^*M$ satisfies

- RA2 The High Order Maximum Principle in the normal form.
- RA3 The Strong Generalized Legendre Condition

$$\mathbb{L} \circ \widehat{\lambda}(t) : \mathbf{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, F_0\}\} \circ \widehat{\lambda}(t) < 0, \qquad t \in [0, \widehat{T}]$$

$$\Downarrow$$

$$\widehat{\mathbf{u}} \in C^\infty([0, \widehat{T}], \mathbb{R}^m)$$

Regularity assumptions I

The adjoint convector $\widehat{\lambda} : [0, \widehat{T}] \to T^*M$ satisfies

- **RA2** The High Order Maximum Principle in the normal form.
- **RA3** The Strong Generalized Legendre Condition

$$\mathbb{L} \circ \widehat{\lambda}(t) : \mathbf{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, F_0\}\} \circ \widehat{\lambda}(t) < 0, \qquad t \in [0, \widehat{T}]$$

$$\Downarrow$$

$$\widehat{\mathbf{u}} \in C^{\infty}([0, \widehat{T}], \mathbb{R}^m)$$

Regularity assumptions I

The adjoint convector $\widehat{\lambda} : [0, \widehat{T}] \to T^*M$ satisfies

- RA2 The High Order Maximum Principle in the normal form.
- RA3 The Strong Generalized Legendre Condition

$$\mathbb{L} \circ \widehat{\lambda}(t) : \mathbf{v} \mapsto \sum_{i,j=1}^{m} \mathrm{v}_i \mathrm{v}_j \{F_i, \{F_j, F_0\}\} \circ \widehat{\lambda}(t) < 0, \qquad t \in [0, \widehat{T}]$$

$$\Downarrow$$

$$\widehat{\mathbf{u}} \in C^\infty([0, \widehat{T}], \mathbb{R}^m)$$

Regularity assumptions II

Fundamental sub-manifolds of T^*M

- $\Sigma = \{\ell \in T^*M \colon F(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$
- $\mathcal{S} = \{\ell \in \Sigma \colon \{F_0, F\}(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$

$\widehat{\lambda}(t)$ is in $\mathcal{S}$ ($\Longleftarrow$ HOMP)

properties coming from RA1–RA3

- the codimension of Σ is r
- $\vec{F}_0$ is tangent to Σ only on $\mathcal{S}$
- Lie$\{\vec{F}_1, \ldots, \vec{F}_m\}$ is tangent to Σ
- $\vec{F}_1, \ldots, \vec{F}_m$ are transversal to $\mathcal{S}$ thus codim $\mathcal{S}$ in Σ is $\geq m$

RA4 The codimension of $\mathcal{S}$ in Σ is $= m$

Regularity assumptions II

Fundamental sub-manifolds of T^*M

- $\Sigma = \{\ell \in T^*M \colon F(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$
- $\mathcal{S} = \{\ell \in \Sigma \colon \{F_0, F\}(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$

$\widehat{\lambda}(t)$ is in $\mathcal{S}$ ($\Longleftarrow$ HOMP)

properties coming from RA1–RA3

- the codimension of Σ is r
- $\vec{F}_0$ is tangent to Σ only on $\mathcal{S}$
- Lie$\{\vec{F}_1, \ldots, \vec{F}_m\}$ is tangent to Σ
- $\vec{F}_1, \ldots, \vec{F}_m$ are transversal to $\mathcal{S}$ thus codim $\mathcal{S}$ in Σ is $\geq m$

RA4 The codimension of $\mathcal{S}$ in Σ is $= m$

Regularity assumptions II

Fundamental sub-manifolds of T^*M

- $\Sigma = \{\ell \in T^*M \colon F(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$
- $\mathcal{S} = \{\ell \in \Sigma \colon \{F_0, F\}(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$

$\widehat{\lambda}(t)$ is in $\mathcal{S}$ ($\Longleftarrow$ HOMP)

properties coming from RA1–RA3

- the codimension of Σ is r
- $\vec{F}_0$ is tangent to Σ only on $\mathcal{S}$
- Lie$\{\vec{F}_1, \dots, \vec{F}_m\}$ is tangent to Σ
- $\vec{F}_1, \dots, \vec{F}_m$ are transversal to $\mathcal{S}$ thus codim $\mathcal{S}$ in Σ is $\geq m$

RA4 The codimension of $\mathcal{S}$ in Σ is $= m$

Regularity assumptions II

Fundamental sub-manifolds of T^*M

- $\Sigma = \{\ell \in T^*M \colon F(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$
- $\mathcal{S} = \{\ell \in \Sigma \colon \{F_0, F\}(\ell) = 0\,,\ \forall\, f \in \mathcal{L}\}$

$\widehat{\lambda}(t)$ is in $\mathcal{S}$ ($\Longleftarrow$ HOMP)

properties coming from RA1–RA3

- the codimension of Σ is r
- $\vec{F}_0$ is tangent to Σ only on $\mathcal{S}$
- Lie$\{\vec{F}_1, \ldots, \vec{F}_m\}$ is tangent to Σ
- $\vec{F}_1, \ldots, \vec{F}_m$ are transversal to $\mathcal{S}$ thus codim $\mathcal{S}$ in Σ is $\geq m$

RA4 The codimension of $\mathcal{S}$ in Σ is $= m$

Hamiltonian approach

The hamiltonian approach consists in lifting trajectories to the cotangent bundle and compare the costs there

- a super-Hamiltonian (possibly time–dependent) H_t s.t.

$$H_t \geq F^{\max}, \quad H_t \circ \hat{\lambda}(t) = F^{\max} \circ \hat{\lambda}(t)$$

- a suitable horizontal Lagrangian sub–manifold Λ s.t.

$\Lambda = \{\ell = d\alpha(x),\ x \in M\}$

$\pi\mathcal{H}_t \colon \Lambda \to M$ loc. invertible

$$[0, \widehat{T}] \times \Lambda \xrightarrow{\ \mathrm{id} \times \mathcal{H}\ } [0, \widehat{T}] \times T^*M$$

$$[0, \widehat{T}] \times T^*M \xrightarrow{\ \mathrm{id} \times \pi\ } [0, \widehat{T}] \times M$$

$$[0, \widehat{T}] \times M \xrightarrow{\ (\mathrm{id} \times \pi\mathcal{H})^{-1}\ } [0, \widehat{T}] \times \Lambda$$

$\omega = (\mathrm{id} \times \mathcal{H})^*(pdq - H_t dt)$ is exact on $[0, \widehat{T}] \times \Lambda \Rightarrow \oint \omega = 0$

Leading ideas

- Use the regularity conditions to define H_t.
- Require the coercivity of a suitable second order approximation. This allows to define Λ and to prove that $\pi\mathcal{H}_{t|\Lambda}$ is locally invertible.

The idea is to choose Λ on Σ and to define an H_t whose flow is tangent to Σ

RA1–RA4 imply that we can define H_0 s.t.

$$H_0(\ell)\begin{cases} \geq F^{\max} & \text{on } \Sigma \\ = F^{\max} & \text{on } \mathcal{S} \end{cases} \qquad \text{and} \qquad \vec{H}_0 \text{ is tangent to } \Sigma$$

The super-Hamiltonian

H_t is obtained by modifying the reference Hamiltonian

$$H_t = H_0 + \sum_{i=1}^{m} \widehat{u}_i(t) F_i \qquad (m = m_1 \text{ in the partially singular case})$$

2^{nd} Order Condition - Minimum Time Problem

pullback vector fields

define: $g_t^i = \widehat{S}_{t*}^{-1} f_1^i \circ \widehat{S}_t, \quad \Longrightarrow \quad \dot{g}_t^i = \widehat{S}_{t*}^{-1}[f_0, f_i] \circ \widehat{S}_t, \qquad i = 1, \ldots, m$

the Hilbert space

Let $\mathcal{W}$ be the subspace of the Hilbert space $T_{\widehat{\xi}(0))} M \times L^2([0, \widehat{T}], \mathbb{R}^m)$ defined by

$$\dot{\zeta}(t) = \sum_{i=1}^{m} w_i(t) \dot{g}_t^i(\widehat{\xi}(0)), \qquad \zeta(0) = \delta x \in \mathcal{L}(\widehat{\xi}(0)) = T_{\widehat{\xi}(0)} I_{\widehat{\xi}(0)}, \qquad \zeta(\widehat{T}) = 0$$

A5 - second order condition

the quadratic form

$$J'' \colon (\delta x, \boldsymbol{w}) \mapsto \frac{1}{2} \sum_{i=1}^{m} \left(\int_0^{\widehat{T}} 2 w_i(t) L_{\zeta(t)} L_{\dot{g}_t^i} \beta(\widehat{\xi}(0)) + \sum_{j=1}^{m} w_i(t) w_j(t) R_{ij}(t) \, dt \right)$$

is coercive on $\mathcal{W}$

The result

If all the regularity assumptions RA1,RA2,RA3, RA4 hold true and the second order approximation is coercive, i.e. A5 is satisfied

then

- $\widehat{\xi}$ is a strong-local optimizer for the minimum-time problem between $I_{\widehat{\xi}(0)}$ and $I_{\widehat{\xi}(\widehat{T})}$
- $\widehat{\xi}$ is strictly optimal between $\widehat{\xi}(0)$ and $I_{\widehat{\xi}(\widehat{T})}$ and between $I_{\widehat{\xi}(0)}$ and $\widehat{\xi}(\widehat{T})$

for the partially singular case $I_{\widehat{\xi}(0)}$ and $I_{\widehat{\xi}(\widehat{T})}$ are integral manifolds of the Lie - algebra defined by the subsystem

The result

If all the regularity assumptions RA1,RA2,RA3, RA4 hold true and the second order approximation is coercive, i.e. A5 is satisfied

then

- $\widehat{\xi}$ is a strong-local optimizer for the minimum-time problem between $I_{\widehat{\xi}(0)}$ and $I_{\widehat{\xi}(\widehat{T})}$
- $\widehat{\xi}$ is strictly optimal between $\widehat{\xi}(0)$ and $I_{\widehat{\xi}(\widehat{T})}$ and between $I_{\widehat{\xi}(0)}$ and $\widehat{\xi}(\widehat{T})$

for the partially singular case $I_{\widehat{\xi}(0)}$ and $I_{\widehat{\xi}(\widehat{T})}$ are integral manifolds of the Lie - algebra defined by the subsystem

The result

If all the regularity assumptions RA1,RA2,RA3, RA4 hold true and the second order approximation is coercive, i.e. A5 is satisfied

then

- $\widehat{\xi}$ is a strong-local optimizer for the minimum-time problem between $I_{\widehat{\xi}(0)}$ and $I_{\widehat{\xi}(\widehat{T})}$
- $\widehat{\xi}$ is strictly optimal between $\widehat{\xi}(0)$ and $I_{\widehat{\xi}(\widehat{T})}$ and between $I_{\widehat{\xi}(0)}$ and $\widehat{\xi}(\widehat{T})$

for the partially singular case $I_{\widehat{\xi}(0)}$ and $I_{\widehat{\xi}(\widehat{T})}$ are integral manifolds of the Lie - algebra defined by the subsystem