The Hamilton-Jacobi problem in nonholonomic mechanics

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Nonholonomic mechanics and optimal control
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A short introduction to Lagrangian mechanics

Let $L = L(q^i, \dot{q}^i)$ be a lagrangian function, where (q^i) are coordinates in a configuration n-manifold Q, and $(\dot{q}^i)$ are the generalized velocities. The Hamilton 's principle produces the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \ 1 \leq i \leq n. \tag{1}$$

A geometric version of Eq. (1) can be obtained as follows.
$L : TQ \longrightarrow \mathbb{R}$. Consider the (1,1)-tensor field S and the Liouville vector field Δ defined on the tangent bundle TQ of Q:

$$S = \frac{\partial}{\partial \dot{q}^i} \otimes dq^i, \ \Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

We construct the Poincaré-Cartan 1 and 2-forms

$$\alpha_L = S^*(dL), \ \omega_L = -d\alpha_L,$$

where S^* denotes the adjoint operator of S. The energy is given by $E_L = \Delta(L) - L$, so that we recover the classical expressions

$$\omega_L = dq^i \wedge dp_i, \ E_L = \dot{q}^i p_i - L,$$

We say that L is regular if the 2-form ω_L is symplectic, which in coordinates turns to be equivalent to the regularity of the Hessian matrix of L with respect to the velocities

$$\left(W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$$

In this case, the equation

$$i_X \, \omega_L = dE_L \qquad (2)$$

has a unique solution, $X = \xi_L$, called the Euler-Lagrange vector field; ξ_L is a second order differential equation (SODE) that means that its integral curves are tangent lifts of their projections on Q (these projections are called the solutions of ξ_L). The solutions of ξ_L are just the ones of Eqs (1).
If $\flat_L : TTQ \longrightarrow T^*TQ$ is the musical isomorphism, $\flat_L(v) = i_v\omega_L$, then we have $\flat_L(\xi_L) = dE_L$.

Legendre transformation

The Legendre transformation $FL : TQ \longrightarrow T^*Q$ is a fibred mapping (that is, $\pi_Q \circ FL = \tau_Q$, where $\tau_Q : TQ \longrightarrow Q$ and $\pi_Q : T^*Q \longrightarrow Q$ denote the canonical projections of the tangent and cotangent bundle of Q, respectively) defined by

$$FL(q^i, \dot{q}^i) = (q^i, p_i),$$

L is regular if and only if FL is a local diffeomorphism.
We will assume that FL is in fact a global diffeomorphism (in other words, L is hyperregular) which is the case when L is a lagrangian of mechanical type, say $L = T - V$ where
- T is the kinetic energy defined by a Riemannian metric on Q,
- $V : Q \longrightarrow \mathbb{R}$ is a potential energy.

Hamiltonian description

The hamiltonian counterpart is developed in the cotangent bundle T^*Q of Q. Denote by $\omega_Q = dq^i \wedge dp_i$ the canonical symplectic form, where (q^i, p_i) are the canonical coordinates on T^*Q. The Hamiltonian energy is just $h = E_L \circ FL^{-1}$ and the Hamiltonian vector field is the solution of the symplectic equation

$$i_{X_h}\omega_Q = dh.$$

The integral curves $(q^i(t), p_i(t))$ of X_h satisfies the Hamilton equations. Since $FL^*\omega_Q = \omega_L$ we deduce that ξ_L and X_h are FL-related, and consequently FL transforms the Euler-Lagrange equations into the Hamilton equations.

Non-holonomic mechanics

Consider a disc rolling without slipping in a rough plane.
Let (x, y) be the coordinates of the point of contact of the disc with the ground, ψ the angle between a point fixed in the circle and the point of contact (the angle of rotation), ϕ the angle between the tangent to the disc at the point of contact and the axis x, and θ the angle of inclination of the disc.
The configuration space is $Q = \mathbb{R}^2 \times S^1 \times S^1 \times S^1$. The lagrangian is $L = T - V$ where

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + R^2\dot{\theta}^2 + R^2\dot{\phi}^2 \sin^2\theta) - mR(\dot{\theta}\cos\phi(\dot{x}\sin\phi - \dot{y}\cos\phi) \\ &+ \dot{\phi}\sin\theta(\dot{x}\cos\phi + \dot{y}\sin\phi)) + \frac{1}{2}I_1(\dot{\theta}^2\dot{\phi}^2\cos^2\theta) + \frac{1}{2}I_2(\dot{\psi} + \dot{\phi}\sin\theta)^2 \end{aligned}$$

and

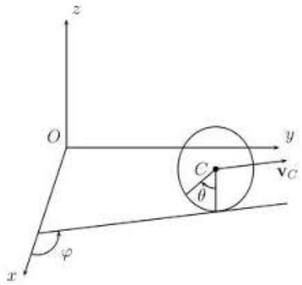
$$V = mgR\cos\theta$$

m is the mass of the disc, R is the radius, I_1 and I_2 are the principle moments of inertia.

The non-slipping condition implies the following constraints.

$$\Phi^1 = \dot{x} - (R\cos\phi)\dot{\psi} = 0, \; \Phi^2 = \dot{y} - (R\sin\phi)\dot{\psi} = 0.$$

All the configurations are possible, but not all the velocities.





Nonholonomic mechanical systems

A nonholonomic mechanical system is given by a lagrangian function $L : TQ \longrightarrow \mathbb{R}$ subject to contraints determined by a linear distribution D on the configuration manifold Q. We will denote by $\mathcal{D}$ the total space of the corresponding vector sub-bundle $(\tau_Q)_{|\mathcal{D}} : \mathcal{D} \longrightarrow Q$ defined by D, where $(\tau_Q)_{|\mathcal{D}}$ is the restriction of the canonical projection $\tau_Q : TQ \longrightarrow Q$.

We will assume that the lagrangian L is defined by a Riemannian metric g on Q and a potential energy $V \in C^\infty(Q)$, so that

$$L(v_q) = \frac{1}{2} g(v_q, v_q) - V(q)$$

or, in bundle coodinates $(q^i, \dot{q}^i)$

$$L(q^i, \dot{q}^i) = \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j - V(q^i)$$

If $\{\mu^a\}$, $1 \leq a \leq k$ is a local basis of the annihilator D^o of D, then the constraints are locally given by

$$\mu_i^a(q)\,\dot{q}^i = 0,$$

where $\mu^a = \mu_i^a(q)\,dq^i$.
The nonholonomic equations can be written as

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} &= \lambda_a \mu_i^a(q) \\ \mu^a(q)\,\dot{q}^i &= 0, \end{aligned}$$

for some Lagrange multipliers λ_a to be determined.
If we modify (11) as follows:

$$i_X\,\omega_L - dE_L \in S^*((T\mathcal{D})^o) \tag{3}$$
$$X \in T\mathcal{D} \tag{4}$$

the unique solution X_{nh} is again a SODE whose solutions are just the ones of the nonholonomic equations.

Let

$$FL : TQ \longrightarrow T^*Q$$

be the Legendre transformation given by

$$FL(q^i, \dot{q}^i) = (q^i, p_i = \frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j)$$

FL is a global diffeomorphism which permits to reinterpret the nonholonomical mechanical system in the hamiltonian side. Indeed, we denote by $h = E_L \circ FL^{-1}$ the hamiltonian function and by $M = FL(\mathcal{D})$ the constraint submanifold of T^*Q.
The nonholonomic equations are then given by

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i} + \bar{\lambda}_a \mu_i^a,$$

where $\bar{\lambda}_a$ are Lagrange multipliers to be determined.

As above, the symplectic equation

$$i_{X_h}\,\omega_Q = dh$$

which gives the hamiltonian vector field X_h should be modified as follows:

$$i_X\,\omega_Q - dh \in F^o \tag{5}$$
$$X \in TM \tag{6}$$

where F is a distribution along M whose annihilator F^o is obtained from $S^*((T\mathcal{D})^o)$ throught FL. Equations (14) and (15) have a unique solution, the nonholonomic vector field X_{nh}.

The Hamilton-Jacobi theory for Hamiltonian systems

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^i)$ (called the **principal function**) such that

$$\frac{\partial S}{\partial t} + h(q^i, \frac{\partial S}{\partial q^i}) = 0, \tag{7}$$

where $h = h(q^i, p_i)$ is the hamiltonian function of the system. If we put $S(t, q^i) = W(q^i) - tE$, where E is a constant, then W satisfies

$$h(q^i, \frac{\partial W}{\partial q^i}) = E; \tag{8}$$

W is called the **characteristic function**.
Equations (7) and (8) are indistinctly referred as the **Hamilton-Jacobi equation**.

The Hamilton-Jacobi equation helps to solve the Hamilton equations for the hamiltonian h

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i} \, , \; \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i} \tag{9}$$

Indeed, if we find a solution W of the Hamilton-Jacobi equation (8) then a solution $(q^i(t))$ of the first set of equations (9) gives a solution of the Hamilton equations by taking $p_i(t) = \frac{\partial W}{\partial q^i}$.

R. Abraham, J.E. Marsden: *Foundations of Mechanics* (2nd edition). Benjamin-Cumming, Reading, 1978.

Let λ be a closed 1-form on Q, say $d\lambda = 0$; (then, locally $\lambda = dW$)

Hamilton-Jacobi Theorem

The following conditions are equivalent:

(i) If $\sigma : I \to Q$ satisfies the equation

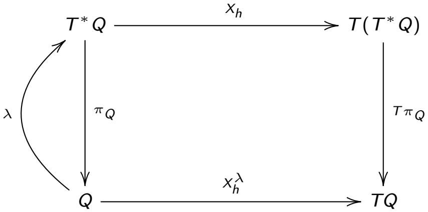
$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(ii) $d(h \circ \lambda) = 0$

Define a vector field on Q:

$$X_h^\lambda = T\pi_Q \circ X_h \circ \lambda$$



The following conditions are equivalent:

(i) If $\sigma : I \to Q$ satisfies the equation

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(i)' If $\sigma : I \to Q$ is an integral curve of X_h^λ, then $\lambda \circ \sigma$ is an integral curve of X_h;

(i)" X_h and X_h^λ are λ-related, i.e.

$$T\lambda(X_h^\lambda) = X_h \circ \lambda$$

Hamilton-Jacobi Theorem
Let λ be a closed 1-form on Q. Then the following conditions are equivalent:

(i) X_h^λ and X_h are λ-related;

(ii) $d(h \circ \lambda) = 0$

If

$$\lambda = \lambda_i(q)\, dq^i$$

then the Hamilton-Jacobi equation becomes

$$h(q^i, \lambda_i(q^j)) = const.$$

and we recover the classical formulation when

$$\lambda_i = \frac{\partial W}{\partial q^i}$$

PROBLEM: How to extend the classical Hamilton-Jacobi theory for nonholonomic mechanical systems?

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G.E. Prince, J.E. Aldridge, G.B. Byrnes: A universal Hamilton-Jacobi equation for second-order ODEs, J. Phys. A: Math. Gen. 32 (1999), 827-844.

V.V. Rumyantsev: Forms of Hamilton's Principle for Nonholonomic Systems, Mechanics, Automatic Control and Robotics 2 (10) (2002), 1035-1048.

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R. Van Dooren: Motion of a rol ling disc by a new generalized Hamilton-Jacobi method, Journal of Applied Mathematics and Physics 27 (1976) 501-505.

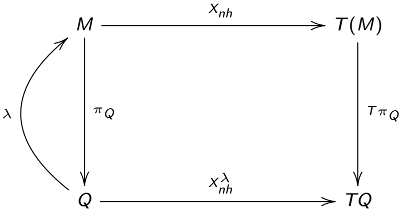
R. Van Dooren: Second form of the generalized Hamilton-Jacobi method for nonholonomic dynamical systems, Journal of Applied Mathematics and Physics 29 (1978) 828-834.

R. Van Dooren: On the generalized Hamilton-Jacobi method for nonholonomic dynamical systems, Dienst Analytische Mechanica, Tw, Vub (1979) 1-6.

Hamilton-Jacobi theory for nonholonomic mechanical systems

Define a vector field on Q:

$$X_{nh}^{\lambda} = T\pi_Q \circ X_{nh} \circ \lambda$$



Let λ be a 1-form on Q taking values into M and satisfying $d\lambda \in \mathcal{I}(D^o)$. Then the following conditions are equivalent:

(i) X_{nh}^{λ} and X_{nh} are λ-related;

(ii) $d(h \circ \lambda) \in D^o$

D. Iglesias, M. de León, D. Martín de Diego: Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems, *J. Phys. A: Math. Theor.* **41** (2008),015205 (14 pp)

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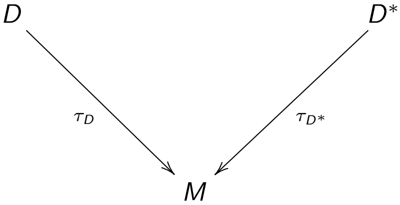
When the system is completely nonholonomic, there is a nice result. Completely nonholonomic means that the iterared Lie brackets

$$[D, D], [D, [D, D]], ...$$

spans the tangent bundle.
Then using the Chows theorem one can prove that there is no non-zero one-form on D^o.

Using Lie algebroids

- $\tau_D : D \longrightarrow M$ a vector bundle, and $\tau_{D^*} : D^* \longrightarrow M$ its dual vector bundle.



- A linear bivector[1] Λ_{D^*} on D^* (not Jacobi identity is required). We denote by $\{\ ,\ \}_{D^*}$ the corresponding almost-Poisson bracket.
- $h : D^* \longrightarrow \mathbb{R}$ a hamiltonian function.

[1] linear means that the bracket of two linear functions is a linear function

The linear bivector Λ_{D^*} induces the following structure on D:

- an almost Lie bracket on the space $\Gamma(\tau_D)$

$$\begin{array}{rccc} [\ ,\ ]_D : & \Gamma(\tau_D)\times\Gamma(\tau_D) & \longrightarrow & \Gamma(\tau_D) \\ & (\xi_1,\xi_2) & \longmapsto & [\xi_1,\xi_2]_D \end{array}$$

  where $\widehat{[\xi_1,\xi_2]_D} = \{\widehat{\xi_1},\widehat{\xi_2}\}_{D^*}$ $([e_\alpha, e_\beta]_D = C^\gamma_{\alpha\beta} e_\gamma)$.
- an anchor map $\rho_D : \Gamma(\tau_D) \longrightarrow \mathfrak{X}(M)$

$$f \in C^\infty(M), \xi \in \Gamma(D) \Rrightarrow \rho_D(\xi)(f)\circ\tau_{D^*} = \{\widehat{\xi}, f\circ\tau_{D^*}\}_{D^*}$$

  (in coordinates, $\rho_D(e_\alpha) = \rho^\mu_\alpha \frac{\partial}{\partial x^\mu}$).

Given local coordinates (x^μ) in the base manifold M and a local basis of sections of D, $\{e_\alpha\}$, we induce local coordinates (x^μ, y_α) on D^*.

Given $\Omega \in \Gamma(\Lambda^k D^*)$ then $d^D\Omega \in \Gamma(\Lambda^{k+1} D^*)$ and

$$\begin{aligned} d^D\Omega(\xi_0, \xi_1, \ldots, \xi_k) &= \sum_{i=0}^{k} (-1)^i \rho_D(\xi_i)(\Omega(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k)) \\ &\quad + \sum_{i<j} \Omega([\xi_i, \xi_j]_D, \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k) \end{aligned}$$

where $\xi_0, \xi_1, \ldots, \xi_k \in \Gamma(\tau_D)$
From the definition, we deduce that

$$\begin{aligned} (1)\ (d^D f)(\xi) &= \rho_D(\xi)(f), \quad f \in C^\infty(M), \quad \xi \in \Gamma(\tau_D) \\ (2)\ d^D\sigma(\xi_1, \xi_2) &= \rho_D(\xi_1)(\sigma(\xi_2)) - \rho_D(\xi_2)(\sigma(\xi_1)) - \sigma[\xi_1, \xi_2]_D, \\ &\quad \sigma \in \Gamma(\tau_{D^*}), \quad \xi_1, \xi_2 \in \Gamma(\tau_D) \\ (3)\ d^D(\Omega \wedge \Omega') &= d^D\Omega \wedge \Omega' + (-1)^k \Omega \wedge d^D\Omega', \\ &\quad \Omega \in \Gamma(\Lambda^k D^*), \Omega' \in \Gamma(\Lambda^{k'} D^*) \end{aligned}$$

In general $\boxed{(d^D)^2 \neq 0}$.

A linear bivector Λ_{D^*} on D^*

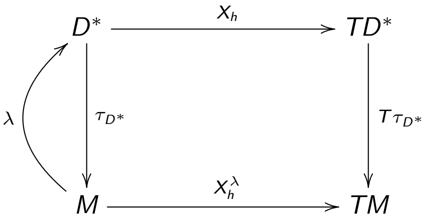
$$\Updownarrow$$

An almost Lie algebroid structure $([\ ,\ ]_D, \rho_D)$ on D

$$\Updownarrow$$

An almost differential $d^D : \Gamma(\Lambda^k D^*) \to \Gamma(\Lambda^{k+1} D^*)$ satisfying (1) and (2)

Let Λ_{D^*} be a linear bivector on D and $\lambda : M \longrightarrow D^*$ be a section of $\tau_{D^*} : D^* \longrightarrow M$



We define $X_h^\lambda = T\tau_{D^*} \circ X_h \circ \lambda$

It is easy to show that $X_h^\lambda(x) \in \rho_D(D_x)$, $\forall x \in M$

Hamilton-Jacobi Theorem

Assume that $d^D\lambda = 0$. Then

$$(i)\ \sigma : I \to M \text{ integral curve of } X_h^\lambda \Rightarrow \lambda \circ \sigma \text{ integral curve of } X_h$$

$$\Updownarrow$$

$$(ii) \quad d^D(h \circ \lambda) = 0$$

Application: Mechanical systems with nonholonomic constraints

Let $\mathcal{G} : E \times_M E \to \mathbb{R}$ be a bundle metric on a Lie algebroid $(E, [\cdot\,,\cdot], \rho)$
The class of systems that were considered is that of mechanical systems with nonholonomic constraints determined by

- The Lagrangian function L:

$$L(a) = \frac{1}{2}\mathcal{G}(a, a) - V(\tau(a)), \quad a \in E,$$

  with V a function on M
- The nonholonomic constraints determined by a subbundle D of E

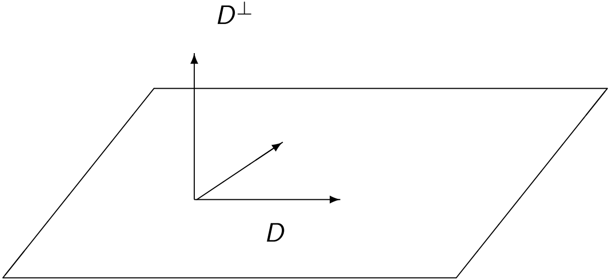
Consider the orthogonal decomposition $E = D \oplus D^\perp$, and the associated orthogonal projectors

$$P : E \longrightarrow D$$
$$Q : E \longrightarrow D^\perp$$

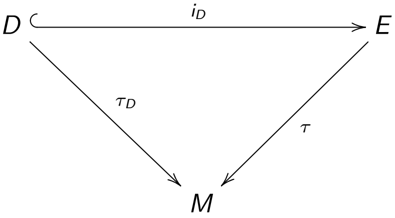
Take local coordinates (x^μ) in the base manifold M and a local basis of sections of E (moving basis), $\{e_\alpha\}$, adapted to the nonholonomic problem (L, D), in the sense that

(i) $\{e_\alpha\}$ is an orthonormal basis with respect to $\mathcal{G}$ (that is $\mathcal{G}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$)

(ii) $\{e_\alpha\} = \{e_a, e_A\}$ where $D = \text{span}\{e_a\}$, $D^\perp = \text{span}\{e_A\}$.



Denoting by $(x^\mu, y^\alpha) = (x^\mu, y^a, y^A)$ the induced coordinates on E, the constraint equations determining D just read $\boxed{y^A = 0.}$ Therefore we choose (x^μ, y^a) as a set of coordinates on D



In these coordinates we have the inclusion

$$\begin{array}{rccc} i_D : & D & \longrightarrow & E \\ & (x^\mu, y^a) & \longmapsto & (x^\mu, y^a, 0) \end{array}$$

and the dual map

$$\begin{array}{rccc} i_D^* : & E^* & \longrightarrow & D^* \\ & (x^\mu, y_a, y_A) & \longmapsto & (x^\mu, y_a) \end{array}$$

where (x^μ, y_α) are the induced coordinates on E^* by the dual basis of $\{e_\alpha\}$.

Moreover, from the orthogonal decomposition we have that

$$P : \begin{array}{ccc} E & \longrightarrow & D \\ (x^\mu, y^a, y^\alpha) & \longmapsto & (x^\mu, y^a) \end{array}$$

and its dual map

$$P^* : \begin{array}{ccc} D^* & \longrightarrow & E^* \\ (x^\mu, y_a) & \longmapsto & (x^\mu, y_a, 0) \end{array}$$

In these coordinates, the nonholonomic system is given by

i) The Lagrangian $L(x^\mu, y^\alpha) = \frac{1}{2}\sum_\alpha (y^\alpha)^2 - V(x^\mu)$,

ii) The nonholonomic constraints $y^A = 0$.

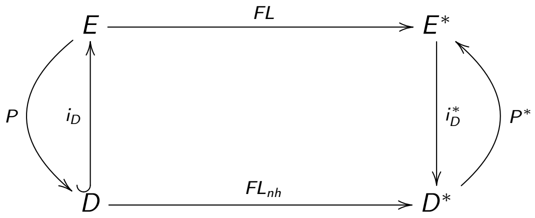
The Legendre transformation associated with L is the isomorphism $FL : E \longrightarrow E^*$ induced by the metric $\mathcal{G}$. Therefore, locally, the Legendre transformation is

$$\begin{array}{rccc} FL: & E & \longrightarrow & E^* \\ & (x^\mu, y^\alpha) & \longmapsto & (x^\mu, y_\alpha = y^\alpha) \end{array}$$

and we can define the nonholonomic Legendre transformation $FL_{nh} = i_D^* \circ FL \circ i_D : D \longrightarrow D^*$

$$\begin{array}{rccc} FL_{nh}: & D & \longrightarrow & D^* \\ & (x^\mu, y^a) & \longmapsto & (x^\mu, y_a = y^a) \end{array}$$

Summarizing, we have the following diagram



The nonholonomic bracket

$(E, [\ ,\ ], \rho)$ is a Lie algebroid

$$\Downarrow$$

Λ_{E^*} is a linear Poisson structure on E^*

If f_1 and f_2 are functions on M, and ξ_1 and ξ_2 are sections of E, then:

$$\{f_1 \circ \tau_{E^*}, g_1 \circ \tau_{E^*}\}_{E^*} = 0, \quad \{\widehat{\xi_1}, f_1 \circ \tau_{E^*},\}_{E^*} = (\rho(\xi_1))\, f_1 \circ \tau_{E^*},$$
$$\{\widehat{\xi_1}, \widehat{\xi_2}\}_{E^*} = \widehat{[\xi_1, \xi_2]}$$

In the induced coordinates (x^μ, y_α), the Poisson bracket relations on E^* are

$$\{x^\mu, x^\eta\}_{E^*} = 0, \qquad \{y_\alpha, x^\mu,\}_{E^*} = \rho^\mu_\alpha, \qquad \{y_\alpha, y_\beta\}_{E^*} = C^\gamma_{\alpha\beta} y_\gamma$$

In other words

$$\Lambda_{E^*} = \rho^\mu_\alpha \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial x^\mu} + \frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}$$

The nonholonomic bracket on D^*, $\{\ ,\ \}_{nh,D^*}$, is defined by

$$\{F, G\}_{nh,D^*} = \{F \circ i_D^*, G \circ i_D^*\}_{E^*} \circ P^*$$

for all $F, G \in C^\infty(D^*)$
The induced bivector Λ_{nh,D^*} is

$$\Lambda_{nh,D^*} = \rho_a^\mu \frac{\partial}{\partial y_a} \wedge \frac{\partial}{\partial x^\mu} + \frac{1}{2} C_{ab}^c y_c \frac{\partial}{\partial y_a} \wedge \frac{\partial}{\partial y_b}$$

That is,

$$\{x^\mu, x^\eta\}_{nh,D^*} = 0, \qquad \{y_a, x^\mu\}_{nh,D^*} = \rho_a^\mu, \qquad \{y_a, y_b\}_{nh,D^*} = C_{ab}^c y_c$$

Λ_{nh,D^*} is a linear bivector on D^*, but in general, does not satisfy Jacobi identity.

Particular cases

1. $E = TM$. Then the linear Poisson structure on $E^* = T^*M$ is the canonical symplectic structure. Thus, D is a distribution on M and $\{\ ,\ \}_{nh,D^*}$ is the nonholonomic bracket studied by A.J. Van der Schaft, B.M. Maschke, and others.
2. $E = \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra. E is a Lie algebroid over a single point (the anchor map is the zero map). In this case, the linear Poisson structure on $E^* = \mathfrak{g}^*$ is the $\pm$ Lie-Poisson structure. Thus, if $D = \mathfrak{h}$ is a vector subspace of $\mathfrak{g}$, we obtain that the nonholonomic bracket (nonholonomic Lie-Poisson bracket) is given by

$$\{F, G\}_{nh,D^*\pm}(\mu) = \pm \left\langle \mu, P\left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle$$

for $\mu \in \mathfrak{h}^*$, and $F, G \in C^\infty(\mathfrak{h}^*)$. In adapted coordinates

$$\{y_a, y_b\}_{nh,D^*\pm} = \pm C_{ab}^c y_c$$

3. $E =$ the Atiyah algebroid associated with a principal G-bundle $\pi : Q \longrightarrow Q/G$

$$E = TQ/G$$

The linear Poisson structure on $E^* = T^*Q/G$ is characterized by the following condition: the canonical projection $T^*Q \longrightarrow T^*Q/G$ is a Poisson epimorphism
(See J.P. Ortega and T. S. Ratiu: Momentum maps and Hamiltonian reduction, Progress in Math., 222 Birkhauser, Boston 2004)
D a G-invariant distribution on Q, D/G is a vector subbundle of $E = TQ/G$
Thus, we obtain a reduced non-holonomic bracket $\{\ ,\ \}_{nh,D^*/G}$ (the non-holonomic Hamilton-Poincare bracket on D^*/G)

We return to the general case

Taking the hamiltonian function $H : E^* \longrightarrow \mathbb{R}$ defined by

$$H(x^\mu, y_\alpha) = \frac{1}{2}\sum_\alpha (y_\alpha)^2 + V(x^\mu)$$

then we induce a hamiltonian function $h : D^* \longrightarrow \mathbb{R}$ by taking $h = H \circ P^*$. In coordinates,

$$h(x^\mu, y_a) = \frac{1}{2}\sum_a (y_a)^2 + V(x^\mu)$$

The nonholonomic dynamics is determined on D^* by the linear bivector Λ_{nh,D^*} and the hamiltonian function $h : D^* \longrightarrow \mathbb{R}$, that is

$$\dot{F} = \{F, h\}_{nh,D^*}$$

So we can apply Hamilton-Jacobi theory to nonholonomic mechanics!

A pure Poisson approach

Let (E, Λ) be an almost Poisson manifold, that is, Λ is a $(2, 0)$ tensor field on a manifold E. We denote by

$$\sharp : T^*E \longrightarrow TE$$

the mapping defined by

$$\langle \sharp(\alpha), \beta \rangle = \Lambda(\alpha, \beta)$$

We denote by $\mathcal{C}$ the characteristic distribution defined by Λ, that is

$$\mathcal{C}(p) = \sharp(T_p^*E)$$

for all $p \in E$.
Notice that $\mathcal{C}$ is a generalized distribution that is not (in general) integrable since Λ is not Poisson in principle.

Definition

A submanifold N of E is said to be a lagrangian submanifold if the following equality holds

$$\sharp(TN^\circ) = TN \cap \mathcal{C}$$

Assume that $\pi : E \longrightarrow M$ is a fibration of E over a manifold M. So we have the diagram

$$\begin{array}{c} (E, \Lambda) \\ \Big\downarrow{\scriptstyle \pi} \\ M \end{array}$$

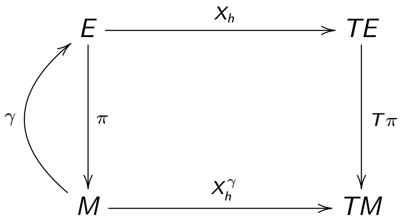
To have dynamics we need to consider a hamiltonian function $h : E \longrightarrow \mathbb{R}$, and thus we obtain the corresponding hamiltonian field

$$X_h = \sharp(dh)$$

Assume that γ is a section of $\pi : E \longrightarrow M$, i.e. $\pi \circ \gamma = id_M$. We can use γ to project X_h on M just defining a vector field X_h^γ on M by

$$X_h^\gamma = T\pi \circ X_h \circ \gamma$$

The following diagram summarizes the above construction:



Assume that $\text{Im}(\gamma)$ is a lagrangian submanifold. Then we have the following theorem.

Theorem

[Hamilton-Jacobi] Under the previous assumptions, the following conditions are equivalent

1. $dh \in (T\,Im(\gamma) \cap \mathcal{C})^{\circ}$.
2. X_h *and* X_h^{γ} *are* γ*-related.*

Assume that (E, Λ) is a transitive Poisson manifold, that is, $\mathcal{C} = TE$. Then, we have

Proposition

A submanifold N of E is a lagrangian submanifold if and only if

$$\sharp(TN^\circ) = TN$$

Note: Symplectic and cosymplectic manifolds are transitive Poisson manifolds.

Therefore, the above theorem takes this form.

Theorem

Assume that $\gamma(M)$ is a lagrangian submanifold of (E, Λ). Then the following assertions are equivalent:

1. X_h *and* X_h^γ *are* γ*-related;*
2. $d(h \circ \gamma) = 0.$

Computations in local coordinates

Assume that (x^i, y^a) are local coordinates adapted to the fibration $\pi : E \longrightarrow M$, that is, $\pi(x^i, y^a) = (x^i)$, where (x^i) are local coordinates in M.

Proposition

$\gamma(M)$ is a lagrangian submanifold of (E, Λ) if and only if

$$\Lambda^{ab} + \Lambda^{bj}\frac{\partial \gamma^a}{\partial x^j} - \Lambda^{aj}\frac{\partial \gamma^b}{\partial x^j} + \Lambda^{ij}\frac{\partial \gamma^a}{\partial x^i}\frac{\partial \gamma^b}{\partial x^j} = 0 \qquad (10)$$

Classical hamiltonian systems

A classical hamiltonian system is given by a hamiltonian function h defined on the cotangent bundle T^*Q of the configuration manifold Q. In this case, $E = T^*Q$ and Λ is the canonical Poisson structure Λ_Q on T^*Q provided by the canonical symplectic form ω_Q on T^*Q.
Recall that now we can take bundle coordinates (q^i, p_i) where $\pi_Q(q^i, p_i) = (q^i)$, and $\pi_Q : T^*Q \longrightarrow Q$ is the canonical projection.
Since in bundle coordinates

$$\omega_Q = dq^i \wedge dp_i$$

we deduce that

$$\Lambda_Q = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_j}$$

Therefore,

$$X_h = \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial}{\partial p_i}$$

and if a section $\gamma : Q \longrightarrow T^*Q$ (that is, a 1-form on Q) is locally expressed by

$$\gamma(q^i) = (q^i, \gamma_i(q^i))$$

we obtain

$$X_h^\gamma = \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q^i}$$

The notion of lagrangian submanifold defined in the almost-Poisson setting reduces to the well-known in the symplectic setting, that is, it is isotropic and coisotropic with respect to the symplectic form ω_Q.
If we compute the condition (10) in this case we obtain

$$\frac{\partial \gamma_i}{\partial q^j} = \frac{\partial \gamma_j}{\partial q^i}$$

which just means that γ is a closed form, i.e., $d\gamma = 0$. So we recover the classical result.

Proposition

Given a 1-form γ on Q, we have that $\gamma(Q)$ is a lagrangian submanifold of (T^*Q, Λ_Q) if and only if γ is closed.

As a consequence, we deduce the classical result.

Theorem

Let γ be a closed 1-form on Q. Then the following assertions are equivalent:

1. *X_h and X_h^γ are γ-related;*
2. *$d(h \circ \gamma) = 0$.*

Nonholonomic mechanical systems

A nonholonomic mechanical system is given by a lagrangian function $L : TQ \longrightarrow \mathbb{R}$ subject to contraints determined by a linear distribution D on the configuration manifold Q. We will denote by $\mathcal{D}$ the total space of the corresponding vector sub-bundle $(\tau_Q)_{|\mathcal{D}} : \mathcal{D} \longrightarrow Q$ defined by D, where $(\tau_Q)_{|\mathcal{D}}$ is the restriction of the canonical projection $\tau_Q : TQ \longrightarrow Q$.
We will assume that the lagrangian L is defined by a Riemannian metric g on Q and a potential energy $V \in C^\infty(Q)$, so that

$$L(v_q) = \frac{1}{2}\, g(v_q, v_q) - V(q)$$

or, in bundle coodinates $(q^i, \dot{q}^i)$

$$L(q^i, \dot{q}^i) = \frac{1}{2}\, g_{ij}\dot{q}^i\dot{q}^j - V(q^i)$$

If $\{\mu^a\}$, $1 \leq a \leq k$ is a local basis of the annihilator D^o of D, then the constraints are locally given by

$$\mu_i^a(q)\,\dot{q}^i = 0,$$

where $\mu^a = \mu_i^a(q)\,dq^i$.
The nonholonomic equations can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \mu_i^a(q)$$
$$\mu^a(q)\,\dot{q}^i = 0,$$

for some Lagrange multipliers λ^i to be determined.

Let S (respectively, Δ) the canonical vertical endomorphism (respectively the Liouville vector field) on TQ. In local coordinates, we have

$$S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i} \; , \; \Delta = \dot{q}^i \, \frac{\partial}{\partial \dot{q}^i}$$

Therefore, we can construct the Poincaré-Cartan 2-form $\omega_L = -dS^*(dL)$ and the energy function function $E_L = \Delta(L) - L$, such that the equation

$$i_{\xi_L} \omega_L = dE_L \tag{11}$$

has a unique solution, ξ_L, which is a SODE on TQ (that is, $S(\xi_L) = \Delta$). Furthermore, its solutions coincide with the solutions of the Euler-Lagrange equations for L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$$

If we modify (11) as follows:

$$i_X \omega_L - dE_L \in S^*((T\mathcal{D})^o) \tag{12}$$

$$X \in T\mathcal{D} \tag{13}$$

the unique solution X_{nh} is again a SODE whose solutions are just the ones of the nonholonomic equations.

Let

$$FL : TQ \longrightarrow T^*Q$$

be the Legendre transformation given by

$$FL(q^i, \dot{q}^i) = (q^i, p_i = \frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j)$$

FL is a global diffeomorphism which permits to reinterpret the nonholonomical mechanical system in the hamiltonian side. Indeed, we denote by $h = E_L \circ FL^{-1}$ the hamiltonian function and by $M = FL(\mathcal{D})$ the constraint submanifold of T^*Q.
The nonholonomic equations are then given by

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i} + \bar{\lambda}_a \mu_i^a,$$

where $\bar{\lambda}^i$ are Lagrange multipliers to be determined.

As above, the symplectic equation

$$i_{X_h}\,\omega_Q = dh$$

which gives the hamiltonian vector field X_h should be modified as follows:

$$i_X\,\omega_Q - dh \in F^o \qquad (14)$$

$$X \in TM \qquad (15)$$

where F is a distribution along M whose annihilator F^o is obtained from $S^*((T\mathcal{D})^o)$ throught FL. Equations (14) and (15) have a unique solution, the nonholonomic vector field X_{nh}.

Another way to obtain X_{nh} is to consider the Withney sum decomposition

$$T(T^*Q)_{|M} = TM \oplus F^{\perp}$$

where the complement is taken with respect to ω_Q. If

$$P : T(T^*Q)_{|M} \longrightarrow TM$$

is the canonical projection onto the first factor, one easily proves that

$$X_{nh} = P(X_h)$$

Moreover, one can introduce an almost-Poisson tensorΛ_{nh} on M by

$$\Lambda_{nh}(\alpha, \beta) = \Lambda_Q(P^*\alpha, P^*\beta)$$

which is called the nonholonomic bracket.
Obviously, we have

$$X_{nh} = \sharp(dh)$$

F. Cantrijn, M. de León, D. Martín de Diego: On almost-Poisson structures in nonholonomic mechanics. *Nonlinearity* **12** (1999), 721–737.

An alternative way to define the nonholonomic bracket is as follows.

L. Bates, J. Sniatycki: Nonholonomic reduction. *Rep. Math. Phys.* **32** (1993), no. 1, 99-115.

Consider the distribution

$$TM \cap F$$

along M. A direct computation shows that the subspace

$$T_pM \cap F_p$$

is symplectic whithin the symplectic vector space $(T_p(T^*Q), \omega_Q(p))$, for all $p \in M$.

Thus we have a second Withney sum decomposition

$$T(T^*Q)_{|M} = (TM \cap F) \oplus (TM \cap F)^\perp$$

where the complement is taken with respect to ω_Q.

If

$$\tilde{P} : T(T^*Q)_{|M} \longrightarrow TM \cap F$$

is the canonical projection onto the first factor, one easily proves that

$$X_{nh} = \tilde{P}(X_h)$$

Moreover, the nonholonomic almost-Poisson tensorΛ_{nh} on M is given by

$$\Lambda_{nh}(\alpha, \beta) = \Lambda_Q(\tilde{P}^*\alpha, \tilde{P}^*\beta)$$

Consider now the fibration

$$\begin{array}{c}(M, \Lambda_{nh}) \\ \Big\downarrow \pi_{Q_{|M}} \\ Q\end{array}$$

and the hamiltonian $h_{|M}$ (also denoted by h in the following). We can easily prove that

$$\mathcal{C}_p = T_pM \cap F_p$$

Indeed,we have

$$\begin{aligned}\langle \sharp_{nh}(\alpha), \beta\rangle &= \omega_Q(\tilde{P}X_\alpha, X_\beta) = -\omega_Q(\beta, \tilde{P}X_\alpha) \\ &= (i_{X_\beta}\,\omega_Q)(\tilde{P}X_\alpha) = -\langle \beta, \tilde{P}X_\alpha\rangle\end{aligned}$$

which implies

$$\sharp_{nh}(\alpha) = -\tilde{P}(X_\alpha)$$

Furthermore, the symplectic structure Ω on $\mathcal{C}_p$ at a point $p \in M$ is given by the restriction of the canonical symplectic structure on T^*Q.

Proposition

Let $\gamma : Q \to M$ be a section of $\pi_{Q|M} : M \longrightarrow Q$, then $\mathrm{Im}(\gamma)$ is a lagrangian submanifold if and only if $d\gamma(X, Y) = 0$ for all $X, Y \in D$.

Proof: We notice that $F = \{v \in T(T^*Q) \text{ such that } T\pi_Q(v) \in D\}$ and an easy computation in local coordinates shows that $\dim(F \cap TM) = 2\dim(D)$. Thus, we have

$$T\mathrm{Im}(\gamma) \cap \mathcal{C} = T\gamma(D)$$

On the other hand, it is clear that our definition of lagrangian submanifold is equivalent to $T\mathrm{Im}(\gamma) \cap \mathcal{C}$ be lagrangian with respect to the simplectic structure Ω on the vector space $\mathcal{C}$. Since Ω is the restriction of Ω_Q, given $X,\ Y \in D$ we have

$$\Omega(T\gamma(X), T\gamma(Y)) = \Omega_Q(T\gamma(X), T\gamma(Y)) = d\gamma(X, Y)$$

So, after a careful counting of dimensions, we deduce that $\mathrm{Im}(\gamma)$ is lagrangian with respect to Λ_{nh} if and only if $d\gamma(X, Y) = 0$ for all $X,\ Y \in D$.

Using this proposition we can recover the Nonholonomic Hamilton-Jacobi Theorem as a consequence of the general Theorem.

Theorem

[Nonholonomic Hamilton-Jacobi] Given a hamiltonian $h : M \to \mathbb{R}$*, and* γ *a* 1*-from on* Q *taking values in* M*, such that* $d\gamma(X, Y) = 0$ *for all* $X,\ Y \in D$*. The following conditions are equivalent*

1. X_h *and* X_h^γ *are* γ*-related.*
2. $dh \in (T\gamma(D))^\circ$ *(which is in turns equivalent to* $d(h \circ \gamma) \in D^\circ$*).*

M. de León, D. Iglesias-Ponte, D. Martín de Diego: Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems. *Journal of Physics A: Math. Gen.* (2008), no. 1, 015205, 14 pp.

M. de León, J.C. Marrero, D. Martín de Diego: Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic mechanics. *J. Geom. Mech.* **2** 2 (2010), 159–198.

T. Oshawa, A.M. Bloch: Nonholonomic Hamilton-Jacobi equations and integrability. *J. Geom. Mech.* **1** 4 (2009), 461–481.

J.F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. Muñoz-Lecanda, N. Román-Roy: Geometric Hamilton-Jacobi theory for nonholonomic dynamical systems. *Int. J. Geom. Meth. Mod. Phys.* **7** 3 (2010), 431–454.

M. Leok, T. Ohsawa, D. Sosa: Hamilton-Jacobi Theory for Degenerate Lagrangian Systems with Holonomic and Nonholonomic Constraints. *J. Math. Phys.* **53** (7) (2012), DOI: 10.1063/1.4736733 .

M. de León, Manuel, D. Martín de Diego, David, M. Vaquero: A Hamilton-Jacobi theory on Poisson manifolds. *J. Geom. Mech.* **6** (2014), 1, 121-140.

We will get a suitable expression for the nonholonomic bracket Λ_{nh} defined on the constraint submanifold M of T^*Q.
Let us recall that the constraints were defined through a distribution D on Q. Let D' a complementary distribution of D in TQ and assume that $\{X_\alpha\}$, $1 \leq \alpha \leq n-k$ is a local basis of D and that $\{Y_a\}$, $1 \leq a \leq k$ is a local basis of D'. Notice that

$$\mu^a(X_\alpha) = 0.$$

Next we introduce new coordinates in T^*Q as follows:

$$\tilde{p}_\alpha = X^i_\alpha p_i \; , \tilde{p}_{n-k+a} = Y^i_a p_i$$

where

$$X_\alpha = X^i_\alpha \frac{\partial}{\partial q^i} \; , \; Y_a = Y^i_a \frac{\partial}{\partial q^i}$$

In these new coordinates we deduce that the constraints become

$$\tilde{p}_{n-k+a} = 0$$

Therefore, we can take local coordinates $(q^i, \tilde{p}_\alpha)$ on M.

A direct computation shows now that the nonholonomic bracket Λ_{nh} on M is given by

$$\Lambda_{nh}(dq^i, dq^j) = 0 \ , \Lambda_{nh}(dq^i, d\tilde{p}_\alpha) = X^i_\alpha$$

$$\Lambda_{nh}(d\tilde{p}_\alpha, d\tilde{p}_\beta) = X^i_\beta p_j \frac{\partial X^j_\beta}{\partial q^i} - X^i_\alpha p_j \frac{\partial X^j_\beta}{\partial q^i}$$

In the sequel, we will apply then general theory developed in Section 2 to the almost-Poisson structure (M, Λ_{nh}).
Assume that $\gamma : Q \longrightarrow M$ is a section of $\pi : M \longrightarrow Q$. Then, we have

$$\gamma(q^i) = (q^i, \tilde{\gamma}_\alpha(q^i))$$

Since γ can also be considered as a 1-form on Q taking values on M we have

$$\gamma(q^i) = (q^i), \gamma_i(q^i))$$

and since it takes values in M we get

$$\tilde{\gamma}_\alpha = X^i_\alpha \, \gamma_i$$

A direct computation from equation (10) gives

$$
\begin{aligned}
&\Lambda_{nh}^{\alpha\beta} + \Lambda_{nh}^{\beta j}\frac{\partial\tilde{\gamma}_\alpha}{\partial q^j} - \Lambda_{nh}^{\alpha j}\frac{\partial\tilde{\gamma}_\beta}{\partial q^j} \\
&= X_\beta^i\gamma_j\frac{\partial X_\alpha^j}{\partial q^i} - X_\alpha^i\gamma_j\frac{\partial X_\beta^j}{\partial q^i} - X_\beta^j\frac{\partial}{\partial q^j}\left(X_\alpha^i\gamma_i\right) - X_\alpha^j\frac{\partial}{\partial q^j}\left(X_\beta^i\gamma_i\right) \\
&= X_\alpha^i X_\beta^j\left(\frac{\partial\gamma_j}{\partial q^i} - \frac{\partial\gamma_i}{\partial q^j}\right) = 0.
\end{aligned}
$$

which can be equivalently written as

$$
d\gamma(X_\alpha, X_\beta) = 0 \tag{16}
$$

Therefore, $\gamma(Q)$ is a lagrangian submanifold of (M, Λ_{nh}) if and if $d\gamma \in \mathcal{I}(D^o)$, where $\mathcal{I}(D^o)$ denotes the ideal of forms generated by D^o. Indeed. notice that (16) holds if and only if $d\gamma = \sum_a \xi_a \wedge \mu^a$, for some 1-forms ξ_a.

Jacobi systems

Let (E, Λ, Z) a Jacobi manifold.
This means that Λ is a bivector field and Z a vector field such that

- $[\Lambda, \Lambda] = 2\, Z \wedge \Lambda$;
- $\mathcal{L}_Z \Lambda = 0$.

Jacobi manifolds include:

- Poisson manifolds
- Locally conformal symplectic manifolds
- contact manifolds

We have the map

$$\sharp : T^*E \longrightarrow TM$$

defined by

$$\langle \sharp(\alpha), \beta \rangle = \Lambda(\alpha, \beta).$$

The hamiltonian vector field associated to a hamiltonian function $h : E \longrightarrow \mathbb{R}$ is

$$X_h = \sharp(dh) + h\,Z$$

The Jacobi bracket is defined by

$$\{f, g\} = \Lambda(df, dg) + fZ(g) - gZ(f)$$

Notice that we do not have the Leibniz rule

$$supp\,\{f, g\} \subset supp\,f \cap supp\,g$$

However

$$[X_f, X_g] = X_{\{f,g\}}$$

The characteristic distribution is now

$$\mathcal{C} = \mathrm{Im}\,\sharp + \langle Z \rangle$$

An $(2n+1)$-dimensional manifold E equipped with a 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

is called a contact manifold.
A contact manifold is a Jacobi manifold equipped with the bivector given by

$$\flat(X) = i_X\, d\eta + \eta(X)\, \eta$$

and the Reeb vector field

$$i_{\mathcal{R}}\, d\eta = 0, i_{\mathcal{R}}\, \eta = 1$$

Definition

A submanifold N of a contact manifold E is said to be a legendrian submanifold if it is an integral manifold of maximal dimension (n) of the distribution $\eta = 0$

We have extended this definition for arbitray Jacobi manifolds as follows.

Definition

A submanifold N of a Jacobi manifold E is said to be a lagrangian-legendrian submanifold if the following equality holds

$$\sharp(TN^{\circ}) = TN \cap \mathcal{C}$$

R. Ibanez, M. de León, J.C. Marrero, D. Martin de Diego: Co-isotropic and Legendre-Lagrangian submanifolds and conformal Jacobi morphisms. *J- Phys. A: Math. Gen.* **30** (1997), 5427–5444.

Everything works if we delete the integrability condition.

Assume that $\pi : E \longrightarrow M$ is a fibration of E over a manifold M. So we have the diagram

$$\begin{array}{c} (E, \Lambda) \\ \Big\downarrow \pi \\ M \end{array}$$

To have dynamics we need to consider a hamiltonian function $h : E \longrightarrow \mathbb{R}$, and thus we obtain the corresponding hamiltonian field

$$X_h = \sharp(dh) + h\,Z$$

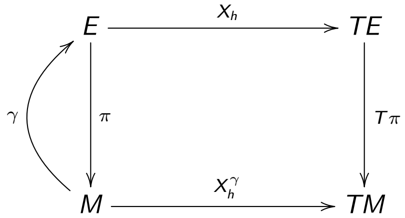
Assume that γ is a section of $\pi : E \longrightarrow M$, i.e. $\pi \circ \gamma = id_M$. We can use γ to project X_h on M just defining a vector field X_h^γ on M by

$$X_h^\gamma = T\pi \circ X_h \circ \gamma$$

Also, we can project Z:

$$Z^\gamma = T\pi \circ Z \circ \gamma$$

The following diagram summarizes the above constructions:



Assume that $\mathrm{Im}(\gamma)$ is a lagrangian-legendrian submanifold of (E, Λ, Z) such that Z and Z^γ are γ-related. Then we have the following theorem.

Theorem

[Hamilton-Jacobi] Under the previous assumptions, the following conditions are equivalent

1. $dh \in (T\,Im(\gamma) \cap \mathcal{C})^\circ$.
2. X_h *and* X_h^γ *are* γ*-related.*