Optimal Control For Differential Inclusions That Have Bounded Variation w.r.t. Time

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Outline of the talk

In optimal control theory:

- Necessary conditions of optimality (e.g. Maximum Principle) are known when **dynamics are measurable w.r.t. time t.**
- Additional properties of minimizing arcs, when dynamics are **regular (constant or Lipschitz cont.) w.r.t. t** :
    - Hamiltonian is constant or Lipschitz continuous
    - necessary conditions are non-degenerate (for extra hyp/s)
    - minimizing arcs have bounded slope, etc.
    - value function is Lipschitz cont.

We show

- Problems where dynamics have **bounded variation w.r.t. t** naturally occur in engineering applications
- Many 'special' properties of arcs for constant or Lipschitz time-dependence generalize to BV time-dependence . . .

The Optimal Control Problem

$$(P)\quad \begin{cases} \text{Minimize}\, g(x(S), x(T)) \\ \text{over } x(.) \in W^{1,1}([S,T], \mathbb{R}^n) \text{ s.t.} \\ \dot{x}(t) \in F(t, x(t)) \text{ a.e.,} \\ h(t, x(t)) \le 0\,, \quad \text{for all } t \in [S,T] \\ (x(S), x(T)) \in C\,, \end{cases}$$

(Data: $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $C \subset \mathbb{R}^n \times \mathbb{R}^n$ (closed) and $F(.,.) : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$.)

Note:

- 'Differential inclusion' formulation
- State constraint '$h(x(t)) \le 0$'

Take a minimizer $\bar{x}(.)$

Hypotheses

(H1): $F(.,.)$ is closed valued, $F(.,x)$ is a $\mathcal{L}$ is measurable for each $x \in \mathbb{R}^n$

(H2): There exist $c > 0$ and $k > 0$ and $\bar{\delta} > 0$ such that

$$F(t,x) \subset F(t,x') + k(|x - x'|)B \quad \text{and} \quad F(t,x) \in c\mathbb{B}\ .$$

for all $x, x' \in \bar{x}(t) + \bar{\delta}B,\ v \in F(t,x)$, a.e. $t \in [S,T]$.

(H3): h(.) is continuously differentiable.

Define Hamiltonian $H(.,.,.) : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$H(t,x,p) := \sup_{v \in F(t,x)} p \cdot v$$

The Hamiltonian Inclusion

Theorem (Measurable Time Dependence). Take a minimizer $\bar{x}(.)$. Assume $(H1)$-$(H3)$.

Then there exist $p(.) \in W^{1,1}$, a $\mathcal{B}$ measure $\mu(.)$ on $[S, T]$, and $\lambda \geq 0$ such that

(i): $\text{supp}\{\mu\} \subset \{t \,|\, h(\bar{x}(t)) = 0\}$

(ii) $(p(.), \lambda, \mu(.)) \neq (0, 0, 0)$,

(iii) $(-\dot{p}(t), \dot{\bar{x}}(t)) \in \text{co}\, \partial_{x,p} H(t, \bar{x}(t), q(t))$ a.e. ,

(iv) $(q(S), -q(T)) \in \lambda \partial g(\bar{x}(S), \bar{x}(T)) + N_C(\bar{x}(S), \bar{x}(T))$,

where $q(t) = \begin{cases} p(S) & \text{if } t = S \\ p(t) + \int_{[S,t]} \nabla h(\bar{x}(s))\mu(ds) & \text{if } t \in (S, T] \end{cases}$

($\partial_{x,p} H(t, x, p)$ is 'subdifferential' of $H(t, .,)$)

(Gives Pontryagin MP when $F(t, x) = f(t, x, U)$.)

Regular Time Dependence: Nec. Conditions

- $F(t,x)$ is indep. of $t \implies H[t] =$ const. **on open interval** (S,T)
- $F(.,x)$ is Lipschitz cont. $\implies t \to H[t]$ is Lipschitz **on open interval** (S,T)

where $H[t]$ is $H(.,.,.)$ evaluated along $(t, \bar{x}(t), q(t))$:

$$H[t] := H(t, \bar{x}(t), q(t))\,.$$

Not obvious because

$$H[t] = \sup_{v \in F(t,\bar{x}(t))} \left\{ \left(p(t) + \int_{[S,t]} \nabla h(\bar{x}(t))\mu(ds) \right) \cdot v \right\}$$

and $\mu(.)$ may have jumps!

Idea of Proof

By considering transformation of the independent variable

$$\sigma(s) = \int_{[S,t]} (1 + w(s))ds,\; w(s) \in [1-\epsilon, 1+\epsilon]$$

show that $(\bar{x}(s), \bar{z}(s) = s)$ is minimizer for an autonomous problem with dynamics

$(\dot{x}(s), \dot{z}(s)) \in \{((1+w)v, (1+w)) \,|\, v \in F(z(s), x(s))$ and $w \in [-\epsilon, +\epsilon]\}$

- Richer class of variations (perturb state trajectories also by 'scaling' time variable) yield **extra information**:

$$H(t, \bar{x}(t, q(t)) = r(s) \quad \text{for } t \in (S, T) \qquad (1)$$

  for some Lipschitz continuous function $r(.)$ satisfying

$$\dot{r} \in \partial_t H(t, \bar{x}(t, q(t))$$

- additional analysis to extend to (1) to all $[S, T]$.
- $F(t, x)$ must be Lipschitz continuous in both variables, because time is now a state variable.

Regularity of the Hamiltonian: Open Questions

Recall

- $F(t, x)$ is independent of $t \implies H[t] = c$
- $F(., x)$ is Lipschitz $\implies t \to H[t]$ is Lipschitz cont.

Interpretation:

'The Hamiltonian $H[t]$ (along the extremal $\bar{x}(.), p(.)$) inherits the time-regularity properties of the dynamics'

Does the Hamiltonian inherit other forms of continuity?

$$\text{'}t \to F(., x) \textbf{ is continuous'} \overset{?}{\implies} \textbf{'Hamiltonian is continous?'}$$

We answer related questions . .

Why is Regularity of the Hamiltonian Useful?

- Lagrangian mechanics constancy of Hamiltonian gives **invariants of motion**. $x(.)$ moves in a conservative force field ($F(x) = \nabla\phi(x)$). Motion $\bar{x}(.)$ minimizes the action

$$-\int \left(\phi(x(t)) - \frac{1}{2}\dot{x}^2(t) \right) dt$$

  Hamiltonian is $\phi(\bar{x}(t)) + \frac{1}{2}\dot{\bar{x}}^2(t)$ **(conservation of energy)**
- Optimality conditions on singular arcs
- Conditions for regularity of optimal controls
- Existence of non-degenerate multipliers

Functions of Bounded Variation

Classical concept:

$r(.) : [S, T] \to \mathbb{R}$ has bounded variation means

$$\eta(T) \; < \; +\infty$$

in which

$$\eta(t) := \sup_{\mathcal{T}}\{ \sum_{i=0}^{N-1} |r(t_{i+1} - r(t_i) \,|\}$$

(Sup taken over all partitions $\mathcal{T} = \{t_0 = S, \ldots, t_N = t\}$ of $[S, t]$.)

$\eta(t)$ is called the cummulative variation function

- $\eta(.)$ is monotone increasing
- $\eta(.)$ has a countable number of continuity points
- $\eta(.)$ has everywhere left and right limits
- **$r(.)$ induces 'signed' Borel measure $A \to \int_A r(dt)$**

Generalization to Multifunctions

Take multifunction $F : [S, T] \times R^n \rightsquigarrow \mathbb{R}^n$ and F- trajectory $\bar{x}(.)$.

Several ways to define '$t \to F(t, .)$' has bounded variation

Definition. $t \to F(t, .)$ has bounded variation along $\bar{x}$ if $\eta(T) < +\infty$, where

$$\eta(t) := \sup_{\mathcal{T}} \left\{ \sum_{i=0}^{N-1} \sup \{ d_H(F(t_{i+1}, x), F(t_i, x)) \mid , x \in X \} \right\} .$$

(Supremum over partitions $\mathcal{T} = \{t_0 = S, \ldots, t_N = t\}$ of $[S, t]$).

$$X := \{x(t) \mid x(.) \text{ is state trajectory }, t \in [S, T]\}$$

$\eta(.)$ is called the cummulative variation of $t \to F(t, .)$

Refined definition

Take $F : [S, T] \times R^n \rightsquigarrow \mathbb{R}^n$ and an F-trajectory $\bar{x}(.)$.

Localize definition about the given state trajectory $\bar{x}(.)$:

Definition. $t \to F(t, .)$ has bounded variation along $\bar{x}(.)$ if $\eta(T) < +\infty$, where

$$\eta(t) := \lim_{\delta,\epsilon \downarrow 0} \eta^{\delta}_{\epsilon}(t)$$

and $\eta^{\delta}_{\epsilon}(t) \ :=$

$$\sup_{\mathcal{T}} \left\{ \sum_{i=0}^{N-1} \sup_{x \in \bar{x}([t_i, t_{i+1}] + \delta\mathbb{B})} \{ d_H(F(t_{i+1}, x), F(t_i, x)) \} \,|\, \text{diam}\{\mathcal{T}\} \leq \epsilon \right\} .$$

(Supremum over partitions $\mathcal{T} = \{t_0 = S, \ldots, t_N = t\}$) of $[S, t]$.)

($\eta(.)$ is cummulative variation of $t \to F(t, .)$ along $\bar{x}(.)$).

Precedents: Moreau's Sweeping Processes

Take a closed, convex multifunction $C(.) : [S, T] \to \mathbb{R}^n$.
Sweeping processes are state trajectories for

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) & \text{a.e. } t \geq 0 \\ x(S) = x_0 \end{cases}$$

(Moreau, 1973)

Hypotheses: $\sup_{\mathcal{T}} \left\{ \sum_{i=0}^{N-1} \sup_{v \in C(t_{i+1})} d_{C(t_i)}(v)) \right\} < \infty$.

(Supremum over partitions $\mathcal{T} = \{t_0 = S, \ldots, t_N = t\}$ of $[S, T]$.)

(Early example of use of BV multifunctions)

Properties

Take a multifunction $F(.,.)$ of bounded variation along $\bar{x}(.)$.

Write $\eta(.) =$ cummulative variation function. Then

$$d_H(F(t,\bar{x}(s)), F(s,\bar{x}(s))) \leq \eta(t) - \eta(s)$$

for all $[s,t] \subset [S,T]$ such that $t - s \leq \epsilon$,

(for $\epsilon > 0$ suff. small).

Multifunctions of BV have many 'classical' properties:

(a): For $s \in [S,T)$ and $t \in (S,T]$, one-sided limits exist:

$$F(s^+,\bar{x}(s)) := \lim_{s' \downarrow s} F(s',\bar{x}(s)) \text{ and } F(t^-,\bar{x}(t)) := \lim_{t' \uparrow t} F(t',\bar{x}(t))$$

(b): There exists a countable set $\mathcal{A}$ s.t., for every $t \in (S,T) \backslash \mathcal{A}$,

$$\lim_{t' \to t} d_H(F(t',\bar{x}(t)), F(t,\bar{x}(T))) = 0\,.$$

Special Case: Partial BV Functions

Take a function $t \to f(t,.)$ having BV along $\bar{x}(.)$.

For any sequence of 'fine' partitions $\{t_i^j\}_{i=0}^{N_j}$, define

$$\mu^j = \sum_{i=0}^{N_j-1} \left[f(t_{i+1}^j, \bar{x}(t_i^j)) - \bar{f}(t_i^j, x(t_i^j)) \right] \delta(t - t_i^j)$$

Then there exists a (signed) Borel measure μ on $[S, T]$ s.t.

$$\mu^j \to \mu \quad \text{(weak}^* \text{ in } C^*([S,T])), \qquad \text{i.e.}$$

$$\int_{[S,T]} g^T(t)\, d\mu^j(t) \to \int_{[S,T]} g^T(t)\, d\mu(t) \text{ for every } g(.) \in C([S,T])\,.$$

Interpret $\int g^T(t) f(dt, \bar{x}(t)) := \int g^T(t)\, d\mu(t)$

sensitivity relations . .

Multifunctions having Bounded Variation

Class of multifunctions $t \to F(.,.)$ with bdd. var. (along some $\bar{x}(.)$) is much larger than the class of Lip. multifunctions $t \to F(t,.)$.

Examples of Mutifunctions having bounded variation include:

- $F(.,.)$'s with a finite number of fractional singularities, e.g.

$$F(t,x) = \sum_{i=1}^{N} |t - t_i|^{\frac{1}{2}} \tilde{F}_i(x) \quad (\tilde{F}(.) \text{ 'smooth' })$$

- $F(.,.)$'s with a finite number of interior discontinuities
- $F(.,.)$'s with end-time discontinuities

BUT

some Hölder $t \to F(t,.)$'s do not have bounded variation.

BV Multifunctions in Control Engineering

Two stage control selection (in aeronautic applictions):

- control Choose an (open loop) $u^*(.)$ yields a nominal trajectory
- Choose a secondary control $v(.)$ to corrects errors due to inaccurate model, etc.

$$\begin{cases} \dot{x}(t) = f(t, x(t), u^*(t) + v(t)) \\ v(t) \in V(t) := U - u^*(t)\,. \end{cases}$$

$u^*(.)$ is typically solution of an optimal control problem.

The differential inclusion for designing the v-control is

$$\dot{x}(t) \in F(t, x(t)) := \{f(x(t), u^*(t) + v \,|\, v \in V(t)\}$$

If $u^*(.)$ is bang-bang (e.g. solves minimum time problem), $F(., x)$ will be discontinuous, but might have BV w.r.t. t.

Hamiltonians of Bounded Variation

Theorem (2014, Palladino + Vinter). Take a minimizer $\bar{x}(.)$. Assume

- $F(.,.)$ is convex valued
- $t \to F(t,.)$ has bounded variation along $\bar{x}(.)$ with variation $\eta(.)$.

Then the multipliers $(p(.), \mu(.), \lambda)$ can be chosen

to satisfy the following **additional condition**:

- $|H[t] - H[s]| \leq K \times (\eta(t) - \eta(s))$
  for all intervals $[s,t] \subset [S,T]$.

i.e. 'Hamiltonian has inherits BV property from data, and has same cummulative variation (modulo scaling)'.

Idea of Proof

Approximate (P) by Autonomous Multistage Problem on the partition: $\{t_0 = S, \ldots, t_N = T\}$:

$$(P') \begin{cases} \text{Minimize } g(x(T)) \\ \text{over } x(.) : [S,T] \to \mathbb{R}^n \text{ s.t.} \\ \dot{x}(t) \in \sum_{i=0}^{N-1} F(t_i, x(t))\chi_{[t_i,t_{i+1})}(t) \text{ a.e.} \\ \text{and} \\ h(x(t)) \le 0\,, \quad \text{for all } t \in [S,T] \\ x(S) = x_0,\ x(T) \in C\,, \end{cases}$$

Strengthened necessary conditions for multiprocess problem give:

$$|H(t_i, \bar{x}(t_i^-), q(t_i)) - H(t_i, \bar{x}(t_{i-1}^+), q(t_i))| \le K(\eta(t_i) - \eta(t_{i-1})) \ldots$$

(link between Hamiltonian and cummulative var. fn.!)

1st Application: Calculus of Variations

$$(Q)\begin{cases} \text{Minimize } \int_S^T L(t, x(t), \dot{x}(t))dt \\ \text{over } x(.) \in W^{1,1}([0,1];\mathbb{R}^n) \text{ s.t.} \\ x(S) = x_0 \text{ and, } x(T) = x_1 \ . \end{cases}$$

(Q) has a minimizer $\bar{x}(.)$ when:

(HE): **(i):** $L(., x, v)$ is $\mathcal{L} \times \mathcal{B}^{n \times n}$ measurable and $L(t., ., .)$ is lower semicontinuous for each $t \in [S, T]$.

(ii): $L(t, x, .)$ is convex for each $(t, x) \in \mathbb{R}^n \times \mathbb{R}^n$.

(iii): There exists a convex function $\theta(.) : \mathbb{R}^+ \to \mathbb{R}^+$ and a number α such that $\lim_{r\uparrow\infty} \theta(r)/r = +\infty$, and

$$L(t, x, v) \geq \theta(|v|) - \alpha|x| \text{ for all } (t, x, v) \in [S, T] \times \mathbb{R}^n \times \mathbb{R}^n \, .$$

Question: Do there exist Lipschitz cont. minimizers? (no Lavrentiev phenomenon).

Ball Mizel Example - Non Lipschitz Minimizer

$$\begin{cases} \text{Minimize } \int_0^1 \{r\dot{x}^2(t) + (x^3(t) - t^2)^2 \dot{x}^{14}(t)\}\, dt \\ \text{over } x \in W^{1,1}([0,1]; R) \text{ satisfying} \\ x(0) = 0, \quad x(1) = k. \end{cases}$$

Here, $r > 0$ and $k > 0$ are constants, linked by the relationship

$$r = (2k/3)^{12}(1 - k^3)(13k^3 - 7).$$

$\exists\, \epsilon > 0$ s.t., $\forall\, k \in (1 - \epsilon, 1)$, the arc $\bar{x}(t) := kt^{2/3}$ is unique minimizer.

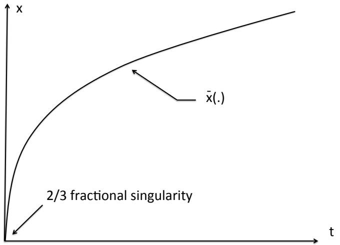

Figure : Non-Lipschitz Minimizer.

Application 1, Continued

(HE) does not guarantee that $\bar{x}(.)$ is Lipschitz. But:

Corollary. Let $\bar{x}(.)$ be a minimizer. Assume that

- (HE)
- $t \to$ epi $L(t,.,.)$ has bounded variation along $(\bar{x}(.), \dot{\bar{x}}(.))$

Then $\bar{x}(.)$ is Lipschitz continuous.

Extends earlier theory: **Replaces** $\exists\, k'(.) \in L^1$ **s.t.**

$\nabla_t L(t, \bar{x}(t), \dot{\bar{x}}(t)) \leq k'(t)$ '$L(., x, v)$ **is Lipschitz** '

(F H Clarke and R B Vinter, 1985)

Proof Technique: Use Tonelli Regularity Theory + strengthened conditions . .

Application 2: Non-degeneracy of Nec. Conds.

If the data is BV then we know

- $|H[t] - H[s]| \leq K \times (\eta(t) - \eta(s))$

for all intervals $[s, t] \subset [S, T]$.

- The strengthened condition can be used to guarantee existence of non-degenerate Lagrange multiplier in some new situations.

Extends earlier theory:

Replace '$F(., x)$ is Lipschitz' by '$F(., x)$ has bounded variation'

Further Applications

- Lipschitz continuity of value functions
- Distance estimates in state constrained optimal control
- Numerical schemes are not ill-conditioned
- New Sensitivity relations
- . . .

New conditions, in which

'$t \to F(t,.)$ **is Lipschitz continuous**'

is replaced by

'$t \to F(t,.)$ **has BV along a given** $\bar{x}(.)$'

Concluding Remarks

- Problems where dynamics have **bounded variation (BV) w.r.t. time** t naturally occur in engineering applications
- Many 'special' properties of arcs for constant or Lipschitz time-dependence generalize to BV time-dependence . . .

'The Hamiltonian inherits the regularity of $t \to F(,.)$'

$t \to F(t,.)$ is constant	$\Longrightarrow$	$H[.]$ is constant
$t \to F(t,.)$ is Lipschitz	$\Longrightarrow$	$H[.]$ is Lipschitz cont.
$t \to F(t,.)$ has BV	$\Longrightarrow$	$H[.]$ has BV (new)

Open Question

$t \to F(t,.)$ is continuous $\Longrightarrow$ Hamiltonian is continuous