Dynamics in systems with constraints

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M $(\dim M = n)$ is the configurational space,

$x = (x_1, \ldots, x_n)$ are the local coordinates,

$\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n)$ are the velocities,

$\Gamma = TM$ is the phase space,

$T(\dot{x}, x, t)$ is the kinetic energy, $\det\left\|\dfrac{\partial^2 T}{\partial \dot{x}^2}\right\| \neq 0$,

$F = (F_1, \ldots, F_n)$ are forces (elements of T^*M).

Equations of motion (Lagrangian equations): $[T] = F$,

$[f] = \dfrac{d}{dt}\dfrac{\partial f}{\partial \dot{x}} - \dfrac{\partial f}{\partial x}$ are Lagrangian derivatives.

Equation of the constraint: $\Phi(\dot{x}, x, t) = 0$, $\dfrac{\partial \Phi}{\partial \dot{x}} \neq 0$.

Usually (in physical applications) $T = \frac{1}{2}(A(x)\dot{x}, \dot{x})$ is positive definite,

$\Phi = (a(x), \dot{x})$ is linear in $\dot{x}$, $a \neq 0$.

How can we determine the motion of the system with constraint?

There are several possibilities.

Nonholonomic model

$$[T] = F + \lambda \frac{\partial \Phi}{\partial \dot{x}}, \quad \Phi = 0 \qquad (NH)$$

Basic principles used to obtain NH:

1°. Release from the constraint: $[T] = F + R$, R is a new force, reaction of the constraint (or the constraint force).

2°. Principle of ideality: $(R, \delta x) = 0$, δx are virtual displacements (variations) of the system with constraint.

$(R, \delta x)$ is the work of the force R on the virtual displacement δx.

3°. Definition of variations (in the state x, $\dot{x}$ at the time moment t):

$$\left(\frac{\partial \Phi}{\partial \dot{x}}, \delta x \right) = 0. \qquad (*)$$

Equivalent principles

A. The d'Alembert–Lagrange principle:

$$([T] - F, \delta x) = 0 \quad \text{for all } \delta x \text{ which satisfy } (*).$$

B. The generalized Hamilton's principle:

$$\delta \int_{t_1}^{t_2} T\, dt + \int_{t_1}^{t_2} (F, \delta x)\, dt = 0$$

for all variations $\delta x(t)$ $(\delta x(t_1) = \delta x(t_2) = 0)$ which satisfy the equation

$$\left(\frac{\partial \Phi}{\partial \dot{x}} \bigg|_{x(t)}, \delta x(t) \right) = 0. \qquad (*)$$

If $F = -\dfrac{\partial V}{\partial x}$, then $L = T - V$ is the Lagrangian.

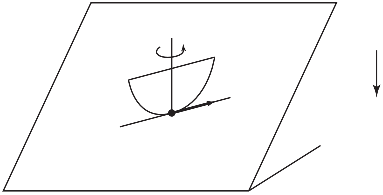
C. The Hölder's principle: an admissible path $t \mapsto x(t)$, $t_1 \leqslant t \leqslant t_2$, is a motion of the given constrained Lagrangian system if and only if it is a critical point (in the sense of Hölder) of the action functional $\displaystyle\int_{t_1}^{t_2} L\, dt$.

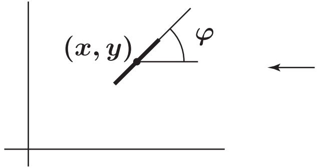
Evidently equation $(*)$ is not equivalent to the equation

$$\delta \Phi = \left(\frac{\partial \Phi}{\partial \dot{x}}, \delta \dot{x} \right) + \left(\frac{\partial \Phi}{\partial x}, \delta x \right) = 0$$

(in general case).

Example: the Chaplygin skate (sleigh)





$$\Phi = \dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$$

$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2) + x$ (for an appropriate choice of mass, length and time units)

$$t = 0: \; x = y = 0, \; \dot{x} = \dot{y} = 0, \; \varphi = 0, \; \dot{\varphi} = \omega$$

$$x = \frac{\sin^2 \omega t}{2\omega^2}, \quad y = \frac{1}{2\omega^2}\left(\omega t - \frac{1}{2}\sin 2\omega t\right), \quad \varphi = \omega t.$$

On the average the skate does not slide down the inclined plane!

Anisotropic friction

$$[L] = -\frac{\partial R}{\partial \dot{x}}, \tag{1}$$

$L = \frac{1}{2}(A(x)\dot{x}, \dot{x}) - V(x)$, R is a Rayleigh's function—a nonnegative definite quadratic form in velocities,

$$(T+V)^{\cdot} = -2R \leqslant 0.$$

Let $R_N = \dfrac{N}{2}(a(x), \dot{x})^2$, where $N > 0$, $a \neq 0$ is a covector field.

Theorem 1. *Let* $x_N(t)$, $t \geqslant 0$, *be a solution of equation* (1) *with an initial condition which does not depend on* N *and satisfies the equation* $(a(x), \dot{x}) = 0$. *Then the limit*

$$\lim_{N\to\infty} x_N(t) = \widehat{x}(t)$$

exists on every bounded time interval $0 \leqslant t \leqslant t_0$. *The limiting function satisfies the system of nonholonomic equations*

$$[L] = \lambda a, \qquad (a, \dot{x}) = 0.$$

C. Carathéodory. ***Z. Angew. Math. Mech.*** **1933. V. 13. P. 71–76.**

V. N. Brendelev. ***J. Appl. Math. Mech.*** **1982. V. 45. no. 3. P. 351–355.**

A. V. Karapetyan. ***J. Appl. Math. Mech.*** **1982. V. 45. no. 1. P. 30–36.**

Servo constraints (by H. Beghin)

Let us consider the free system and suppose that we want to realize the motion with constraint $\Phi = 0$ by using control forces λM. Here M is a fixed covector field (for example, a force field), and $\lambda = \lambda(t)$ is a function to be determined below. We have

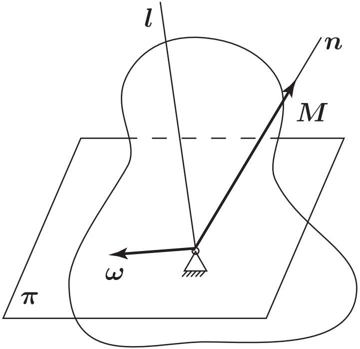
$$[T] = F + \lambda M, \qquad \Phi = 0.$$

The condition of realizability of the constraint $\Phi = 0$ is

$$\left(A^{-1}M, \frac{\partial \Phi}{\partial \dot{x}}\right) \neq 0, \qquad A = \frac{\partial^2 T}{\partial \dot{x}^2}. \qquad (*)$$

Of course, the condition $(*)$ holds if $M = \dfrac{\partial \Phi}{\partial \dot{x}}$ $(\neq 0)$.

Example. **The axes l and n are fixed in the rigid body.**
Constraint: $\omega \perp l$.



M **is the control torque,**
ω **is an angular velocity,**
$l \perp \pi$

Condition (∗) is violated if $l \perp n$. **If** $l = n$ **then we obtain the nonholonomic motion of the rigid body with constraint** $(\omega, l) = 0$ **(Suslov's problem).**

Vakonomic (variational axiomatic kind) model

$$\delta \int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2} (F, \delta x)\,dt = 0$$

for all $\delta x(t)$, $t_1 \leqslant t \leqslant t_2$ ($\delta x(t_1) = \delta x(t_2) = 0$), and

$$\delta \Phi = \left(\frac{\partial \Phi}{\partial \dot{x}}, \delta \dot{x} \right) + \left(\frac{\partial \Phi}{\partial x}, \delta x \right) = 0.$$

Equations of motion:

$$[T] = F + \lambda[\Phi] + \dot{\lambda}\,\frac{\partial \Phi}{\partial \dot{x}}, \qquad \Phi = 0.$$

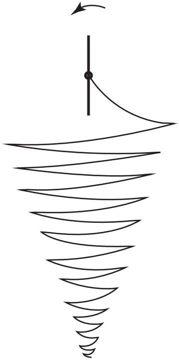
If $F = -\dfrac{\partial V}{\partial x}$ then any motion of vakonomic system is a solution of the Lagrange variational problem.

Example. **The vakonomic skate:**

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2) + x,$$
$$\Phi = \dot{x}\sin\varphi - \dot{y}\cos\varphi.$$

Let $\varphi = 0$ and $\dot{y} + \lambda\cos\varphi = 0$ for $t = 0$. Then

$$\ddot{\varphi} = t^2 \sin\varphi\cos\varphi,$$
$$\dot{x} = t\cos^2\varphi \;\; (> 0 \text{ for } t > 0),$$
$$\dot{y} = t\sin\varphi\cos\varphi$$



Added masses

$$L_N = \frac{1}{2}(A(x)\dot{x}, \dot{x}) + \frac{N}{2}(a(x), \dot{x})^2 - V(q), \quad N > 0.$$

Let $x_N(t)$, $t > 0$, denote the motion of such a system with initial condition x_0, $\dot{x}_0$ satisfying $(a(x_0), \dot{x}_0) = 0$.

THEOREM 2. *The limit*

$$\lim_{N\to\infty} x_N(t) = \widehat{x}(t)$$

exists on every bounded time interval $0 \leqslant t \leqslant t_0$.
The limit $t \mapsto \widehat{x}(t)$ *is an extremal of Lagrange's variational problem*

$$\delta \int_{t_1}^{t_2} L_0\, dt = 0, \quad L_0 = \frac{1}{2}(A\dot{x}, \dot{x}) - V,$$

under the linear constraint $(a, \dot{x}) = 0$.

V. V. Kozlov. I–V. *Mosc. Univ. Mech. Bull.*
37, no. 3–4, p. 27–34 (1982); 37, no. 3–4, p. 74–80 (1982);
38, no. 3, p. 40–51 (1983); 42, no. 5, p. 40–49 (1987);
43, no. 6, p. 23–29 (1988).

Theorem 2 is connected with the method of penalty functions.

Servo constraints

Consider the motion of the system with kinetic energy

$$T_* = T + \lambda N,$$

where N is some function of x, $\dot{x}$ and t, and $\lambda = \lambda(t)$ a function to be determined.

Equations of motion:

$$[T_*] = F, \quad \Phi = 0$$

or

$$[T] = F - \dot{\lambda}\frac{\partial N}{\partial \dot{x}} - \lambda[N], \quad \Phi = 0. \qquad (*)$$

The condition of solvability of this system w.r.t. $\ddot{x}$ and $\dot{\lambda}$ is

$$\left(A^{-1}\frac{\partial N}{\partial \dot{x}}, \frac{\partial \Phi}{\partial \dot{x}}\right) \neq 0. \qquad (**)$$

Let $N = \Phi$. If $\dfrac{\partial \Phi}{\partial \dot{x}} \neq 0$ then condition $(**)$ holds. In this case we obtain the vakonomic motion:

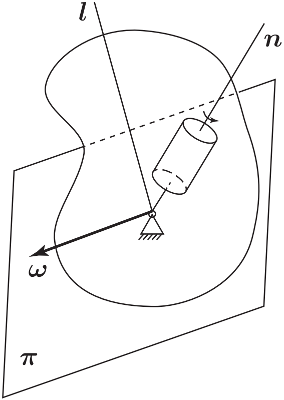
$$\delta \int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2} (F, \delta x)\,dt = 0, \quad \Phi = 0$$

for all variations δx with fixed extremities and

$$\delta \Phi = \left(\frac{\partial \Phi}{\partial \dot{x}}, (\delta x)^{\cdot} \right) + \left(\frac{\partial \Phi}{\partial x}, \delta x \right) = 0.$$

See: V. V. Kozlov. *Mosc. Univ. Mech. Bull.* 1989. № 5. P. 59–66.

Example.



Control angular momentum

$$l \perp \pi$$

Condition (∗∗) is violated if $l \perp n$. If $l = n$ then we obtain the vakonomic motion of the rigid body with constraint $(\omega, l) = 0$. For unique determination of the body motion we must know the initial velocity of the gyroscope as well.

$(*) \implies [T] = F + R, \quad R = -\dot{\lambda}\,\frac{\partial N}{\partial \dot{x}} - \lambda[N]$ is "the reaction of constraint".

This is a version of a release from the constraint.

$$\int_{t_1}^{t_2} (R, \delta x)\, dt = \int_{t_1}^{t_2} (\lambda\, \delta N)\, dt = 0 \quad \text{if} \tag{1}$$

$$\delta N = \left(\frac{\partial N}{\partial \dot{x}}, \delta \dot{x}\right) + \left(\frac{\partial N}{\partial x}, \delta x\right) = 0. \tag{2}$$

(1) is the principle of ideality, and (2) is the definition of variations.