Straight line rolling of an ellipsoid on a plane and the Chasles theorem

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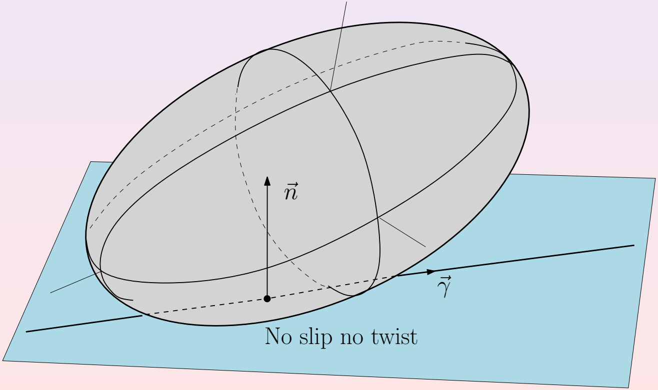
$$Q = \left\{ \frac{X_1^2}{A_1} + \frac{X_1^2}{A_2} + \frac{X_3^2}{A_3} = 1 \right\} \subset \mathbb{R}^3 = (X_1, X_2, X_3)$$



The Jacobi geodesic problem on the ellipsoid Q

Linealization of the geodesic flow on Q (Jacobi and Weierstrass):
Let λ_1, λ_2 be the ellipsoidal coordinates on Q,

$$X_i^2 = A_i \frac{(A_i - \lambda_1)(A_i - \lambda_2)}{(A_i - A_j)(A_i - A_k)}, \qquad i = 1, 2, 3.$$

After time re-parametrization $ds = \lambda_1 \lambda_2 \, ds_1$, reduction to quadratures (Jacobi, 1881)

$$\frac{d\lambda_1}{2\sqrt{R(\lambda_1)}} + \frac{d\lambda_2}{2\sqrt{R(\lambda_2)}} = ds_1,$$

$$\frac{\lambda_1 d\lambda_1}{2\sqrt{R(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{2\sqrt{R(\lambda_2)}} = 0,$$

$$R(\lambda) = -\lambda(\lambda - A_1)(\lambda - A_2)(\lambda - A_3)(\lambda - c)$$

where c is a constant of motion such that the geodesic is tangent to the *caustic* $Q \cap Q_c$.

Geodesic flow on $n-1$-dimensional quadric

Family of confocal quadrics in $\mathbb{R}^n(X_1,\ldots,X_n)$

$$Q(c)=\left\{\frac{X_1^2}{A_1-c}+\cdots+\frac{X_n^2}{A_n-c}=1\right\},\quad c\in\mathbb{R}.$$

Theorem (Chasles) Let $X(s)$ be a geodesic on $Q=Q(0)$ with a natural parameter s and $\vec{\gamma}(s)=dX/ds$ be the tangent vector. Then

1) the tangent line $\ell=\{X+t\vec{\gamma}|t\in\mathbb{R}\}$ is also tangent to $n-1$ fixed confocal quadrics $Q(c_1)=Q(0),Q(c_2),\ldots,Q(c_{n-1})$.

2) Let $\vec{q}_j$ be a unit normal vector of $Q(c_j)$ at the contact point $p_j=\ell\cap Q(c_j)$. The vectors $\vec{q}_1,\ldots,\vec{q}_{n-1},\gamma$ form an *orthogonal* frame in $\mathbb{R}^n$.

Theorem (following J.Moser) When $X(s)$ traces a geodesic on Q, the evolution of $\vec{q}_1, \ldots, \vec{q}_{n-1}, \gamma$ is described by

$$\frac{d}{ds}\vec{q}_j = -\Omega\vec{q}_j, \quad j = 1, \ldots, n-1, \quad \frac{d}{ds}\vec{\gamma} = -\Omega\gamma,$$
$$\Omega = \langle X, A^{-2}X\rangle A^{-1}X \wedge A^{-1}\gamma = \vec{q} \wedge A^{-1}\gamma \,.$$

Corollary. In the reference frame $\mathcal{R} = \{\vec{q}_1, \ldots, \vec{q}_{n-1}, \gamma\}$ the ellipsoid Q rolls on the hyperplane $\mathcal{H} = \mathrm{span}(\vec{q}_1 2, \ldots, \vec{q}_{n-1}, \gamma)$ without slipping and twisting. On $\mathcal{H}$ the contact point $Q \cap \mathcal{H}$ moves alongs the line $\mathcal{L}$ =span($\vec{\gamma}$). On the ellipsoid Q the contact point traces the geodesic $X(s)$.
In the frame $\mathcal{R}$ the angular velocity of Q has the form

$$\bar{\Omega} = \begin{bmatrix} 0 & \Omega_{12} & \cdots & \Omega_{1,n} \\ \Omega_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{1,n} & 0 & \cdots & 0 \end{bmatrix}$$

For $n = 3$ the angular velocity vector of Q satisfies $\langle \bar{\omega}, \vec{q}_1 \rangle = 0$

Relation with the Neumann system (H. Knörrer)

Let $X(s)$ be a geodesic on Q, and $q(s) = q_1(s)$ be the normal unit vector of Q at the point X. Then $q(s_1)$ is a solution of the Neumann problem on $S^{n-1} = \{\langle q, q\rangle\} = 1$ with $H = \frac{1}{2}(\langle \dot{q}, \dot{q}\rangle - \langle q, Iq\rangle)$, $I = A^{-1}$:

$$\frac{d^2}{ds_1^2} q = Iq + \nu q,$$

provided that $ds = \langle X, A^{-2} X\rangle ds_1$.

Separation of variables for the Neumann system

Let $\lambda_1, \dots, \lambda_{n-1}$ be spheroconical coordinates on $S^{n-1} = \{\langle q, q\rangle = 1\}$:

$$q_i^2 = \frac{(I_i - \lambda_1)\cdots(I_i - \lambda_{n-1})}{\prod_{j\neq i}(I_i - I_j)} \,,$$

then

$$\begin{cases} \dfrac{d\lambda_1}{2\sqrt{R(\lambda_1)}} + \cdots + \dfrac{d\lambda_{n-1}}{2\sqrt{R(\lambda_{n-1})}} = 0, \\ \qquad\qquad \dots \qquad \dots \\ \dfrac{\lambda_1^{n-2} d\lambda_1}{2\sqrt{R(\lambda_1)}} + \cdots + \dfrac{\lambda_{n-1}^{n-2} d\lambda_{n-2}}{2\sqrt{R(\lambda_{n-1})}} = ds_1, \end{cases}$$

$$R(\lambda) = -(\lambda - I_1)\cdots(\lambda - I_n)\cdot(\lambda - C_1)\cdots(\lambda - C_{n-1}),$$

$$C_1 = 0, \quad C_j = 1/c_j \quad j = 2, \dots, n-1$$

The rotation matrix of Q in the fixed frame $\mathcal{R} = \{\vec{q}_1, \dots, \vec{q}_{n-1}, \gamma\}$:

$$\mathfrak{R} = (\vec{q}_1 \ \cdots \vec{q}_{n-1} \ \gamma)^T \in SO(n)$$

in terms of points $P_1 = (\lambda_1, \mu_1), \dots, P_{n-1} = (\lambda_{n-1}, \mu_{n-1})$ on the genus $g = n - 1$ hyperelliptic curve

$\Gamma : \mu^2 = -(\lambda - I_1) \cdots (\lambda - I_n) \cdot (\lambda - C_1) \cdots (\lambda - C_{n-1})$, $C_1 = 0$

$$q_{1,i} = \frac{\sqrt{U(I_i)}}{\sqrt{\Psi'(I_i)}}, \quad i = 1, \dots, n,$$

$$q_{s,i} = \frac{\sqrt{U(I_i)}}{\sqrt{\Psi'(I_i)}} \frac{\sqrt{U(C_s)}}{\sqrt{\psi'(C_s)}} \sum_{k=1}^{n-1} \frac{\mu_k}{(I_i - \lambda_k)(C_s - \lambda_k)}, \quad s = 2, \dots, n-1,$$

$$\gamma_i = \frac{\sqrt{U(I_i)}}{\sqrt{\Psi'(I_i)}} \frac{\sqrt{U(0)}}{\sqrt{\psi'(0)}} \sum_{k=1}^{n-1} \frac{\mu_k}{(I_i - \lambda_k)\lambda_k},$$

where

$$U(r) = (r - \lambda_1) \cdots (r - \lambda_{n-1}),$$

$$\Psi(\lambda) = (\lambda - I_1) \cdots (\lambda - I_n), \quad \psi(\lambda) = (\lambda - C_1) \cdots (\lambda - C_{n-1}).$$

(Following Yu. F., B. Jovanović. J. Nonl. Sci. 2004)

The angular velocity of Q in the frame $\mathcal{R} = \{\vec{q}_1, \dots, \vec{q}_{n-1}, \gamma\}$

$$\bar{\Omega} = \begin{bmatrix} 0 & \Omega_{12} & \cdots & \Omega_{1,n} \\ \Omega_{12} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{1,n} & 0 & \cdots & 0 \end{bmatrix}$$

with

$$\Omega_{1,s} = \frac{\sqrt{U(0)}}{\sqrt{\psi'(0)}} \frac{\sqrt{U(C_s)}}{\sqrt{\psi'(C_s)}} \sum_{k=1}^{n-1} \frac{\mu_k}{\lambda_k(C_s - \lambda_k)}, \quad s = 2, \dots, n-1,$$

$$\Omega_{1,n} = \frac{\sqrt{U(0)}}{\sqrt{\psi'(0)}}.$$

The coordinates of the center of Q in $\mathcal{R}$:

$$(\langle X, q_1 \rangle, \cdots, \langle X, q_{n-1} \rangle, s + \langle X, \gamma \rangle),$$

$$X = \sqrt{\lambda_1 \cdots \lambda_{n-1}}\, A q_1, \qquad s = \int \lambda_1 \cdots \lambda_{n-1}\, ds_1.$$

The angular velocity of the frame $\mathcal{R} = \{\vec{q}_1, \dots, \vec{q}_{n-1}, \gamma\}$ with respect to the axes $\{X_1, \dots, X_n\}$ of the ellipsoid Q:

$$\omega_{ij} = \frac{\sqrt{U(I_i)}}{\sqrt{\Psi'(I_i)}} \frac{\sqrt{U(I_j)}}{\sqrt{\Psi'(I_j)}} \sum_{k=1}^{n-1} \frac{\mu_k}{(I_i - \lambda_k)(I_j - \lambda_k)}, \quad 1 \le i < j \le n.$$

Theta-function solution

Let B be the $g \times g$ period matrix of the genus $g = n - 1$ curve

$$\Gamma \ : \ \mu^2 = -(\lambda - I_1) \cdots (\lambda - I_n) \cdot (\lambda - C_1) \cdots (\lambda - C_{n-1}), \quad C_1 = 0.$$

Introduce the corresponding theta-function

$$\theta(z|B) = \sum_{M \in \mathbb{Z}^g} \exp(\langle BM, M \rangle / 2 + \langle M, z \rangle),$$

$$\langle M, z \rangle = \sum_{i=1}^{g} M_i z_i, \quad \langle BM, M \rangle = \sum_{i,j=1}^{g} B_{ij} M_i M_j, \quad z \in \mathbb{C}^g,$$

as well as theta-functions *with characteristics* $\alpha = (\alpha_1, \ldots, \alpha_g)$, $\beta = (\beta_1, \ldots, \beta_g)$, $\alpha_j, \beta_j \in \mathbb{R}$, which are obtained from $\theta(z|B)$ by shifting the argument z and multiplying by an exponent:

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z) = \exp\{\langle B\alpha, \alpha \rangle / 2 + \langle z + 2\pi \jmath \beta, \alpha \rangle\} \, \theta(z + 2\pi \jmath \beta + B\alpha).$$

Theta-function solution (II)

- Half-integer theta-characteristics $\eta_i = [\eta_i'', \eta_i']$ such that

$$2\pi \jmath \, \eta_i'' + B\eta_i' = \int_\infty^{(I_i,0)} \bar{\omega} \quad \text{or} \quad = \int_\infty^{(C_i,0)} \bar{\omega}$$

The rotation matrix of Q in the fixed frame $\mathcal{R}$ is

$$\mathfrak{R} = \begin{pmatrix} \sigma_1 \dfrac{\theta[\eta_{I_1}](z)}{\theta(z)} & \varepsilon_1 \dfrac{\theta[\eta_{I_1} + \eta_{C_2}](z)}{\theta(z)} & \varkappa_1 \dfrac{\theta[\eta_{I_1} + \eta_{C_1}](z)}{\theta(z)} \\ \sigma_2 \dfrac{\theta[\eta_{I_2}](z)}{\theta(z)} & \varepsilon_2 \dfrac{\theta[\eta_{I_2} + \eta_{C_2}](z)}{\theta(z)} & \varkappa_2 \dfrac{\theta[\eta_{I_2} + \eta_{C_1}](z)}{\theta(z)} \\ \sigma_3 \dfrac{\theta[\eta_{I_3}](z)}{\theta(z)} & \varepsilon_3 \dfrac{\theta[\eta_{I_3} + \eta_{C_2}](z)}{\theta(z)} & \varkappa_3 \dfrac{\theta[\eta_{I_3} + \eta_{C_1}](z)}{\theta(z)} \end{pmatrix},$$

$$z = \mathbf{v} s_1 + z_0\,, \quad \mathbf{v} \in \mathbb{C}^n, \quad z_0 = \text{const} \in \mathbb{C}^n.$$